COM/BLM 426
Information Security

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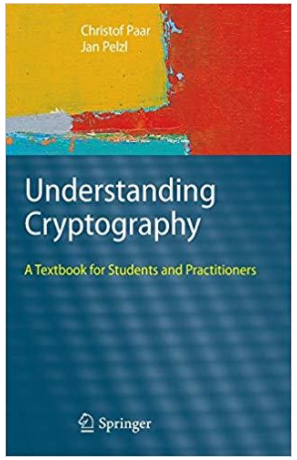
Slides are mainly based on:

Understanding Cryptography: A Textbook for Students and Practitioners



by Christof Paar and Jan Pelzl

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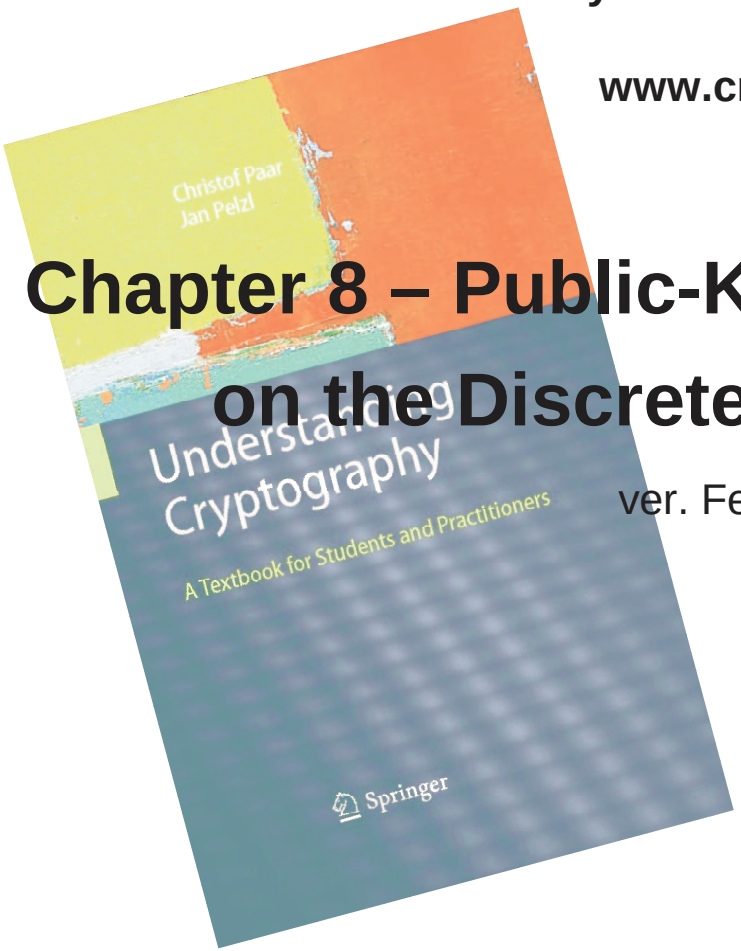
Understanding Cryptography

by Christof Paar and Jan Pelzl

www.crypto-textbook.com



Chapter 8 – Public-Key Cryptosystems Based on the Discrete Logarithm Problem

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These slides were prepared by Christof Paar and Jan Pelzl

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Content of this Chapter

- Diffie–Hellman Key Exchange
- The Discrete Logarithm Problem
- Security of the Diffie–Hellman Key Exchange
- The Elgamal Encryption Scheme

Diffie–Hellman Key Exchange: Overview

- Proposed in 1976 by **Whitfield Diffie and Martin Hellman**
- **Widely used**, e.g. in Secure Shell (SSH), Transport Layer Security (TLS), and Internet Protocol Security (IPSec)
- The Diffie–Hellman Key Exchange (DHKE) is a key exchange protocol and **not** used for encryption

  (For the purpose of encryption based on the DHKE, ElGamal can be used.)

Diffie–Hellman Key Exchange: Set-up

1. Choose a large prime p.
2. Choose an integer $\alpha \in \{2,3, \ldots, p-2\}$.
3. Publish p *and* α.

Diffie–Hellman Key Exchange

Alice		Bob
Choose random private key $k_{prA}=a \in \{1,2,\ldots,p\text{-}1\}$		Choose random private key $k_{prB}=b \in \{1,2,\ldots,p\text{-}1\}$
Compute corresponding public key $k_{pubA}= A = \alpha^a \bmod p$	A →	
	← B	Compute correspondig public key $k_{pubB}= B = \alpha^b \bmod p$
Compute common secret $k_{AB} = B^a = (\alpha^a)^b \bmod p$		Compute common secret $k_{AB} = A^b = (\alpha^b)^a \bmod p$

We can now use the joint key k_{AB} for encryption, e.g., with AES

$$y = AES_{kAB}(x) \xrightarrow{\quad y \quad} x = AES^{-1}_{kAB}(y)$$

Diffie–Hellman Key Exchange: Example

Domain parameters p=29, α=2

Alice		Bob
Choose random private key $k_{prA} = a = 5$		Choose random private key $k_{prB} = b = 12$
Compute corresponding public key $k_{pubA} = A = 2^5 = 3 \bmod 29$	A →	
	← B	Compute correspondig public key $k_{pubB} = B = 2^{12} = 7 \bmod 29$
Compute common secret $k_{AB} = B^a = 7^5 = 16 \bmod 29$		Compute common secret $k_{AB} = A^b = 3^{12} = 16 \bmod 29$

Proof of correctness:

Alice computes: $B^a = (\alpha^b)^a \bmod p$
Bob computes: $A^b = (\alpha^a)^b \bmod p$

i.e., Alice and Bob compute the same key k_{AB} !

The Discrete Logarithm Problem

Discrete Logarithm Problem (DLP) in Z_p^*

- Given is the finite cyclic group Z_p^* of order $p-1$ and a primitive element $\alpha \in Z_p^*$ and another element $\beta \in Z_p^*$.
- The DLP is the problem of determining the integer $1 \leq x \leq p-1$ such that $\alpha^x \equiv \beta \bmod p$
- This computation is called the **discrete logarithm problem (DLP)**

$$x = log_\alpha \beta \bmod p$$

- Example: Compute x for $5^x \equiv 41 \bmod 47$

Remark: For the coverage of groups and cylcic groups, we refer to Chapter 8 of *Understanding Cryptography*

The Generalized Discrete Logarithm Problem

- Given is a finite cyclic group G with the group operation $\circ$ and cardinality n.
- We consider a primitive element $\alpha \in G$ and another element $\beta \in G$.
- The discrete logarithm problem is finding the integer x, where $1 \leq x \leq n$, such that:

$$\beta = \underbrace{\alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha}_{x \text{ times}} = \alpha^x$$

The Generalized Discrete Logarithm Problem

The following discrete logarithm problems have been proposed for use in cryptography

1. The multiplicative group of the prime field Z_p *or a subgroup of it. For instance,* the classical DHKE uses this group (cf. previous slides), but also Elgamal encryption or the Digital Signature Algorithm (DSA).
2. The cyclic group formed by an elliptic curve (see Chapter 9)
3. The multiplicative group of a Galois field $GF(2^m)$ or a subgroup of it. *S*chemes such as the DHKE can be realized with them.
4. Hyperelliptic curves or algebraic varieties, which can be viewed as generalization of elliptic curves.

Remark: The groups 1. and 2. are most often used in practice.

Attacks against the Discrete Logarithm Problem

- Security of many asymmetric primitives is based on the difficulty of computing the DLP in cyclic groups, i.e.,

  Compute x for a given α and β such that $\beta = \alpha \circ \alpha \circ \alpha \circ \ldots \circ \alpha = \alpha^x$

- The following algorithms for computing discrete logarithms exist
  - Generic algorithms: Work in any cyclic group
    - Brute-Force Search
    - Shanks‘ Baby-Step-Giant-Step Method
    - Pollard‘s Rho Method
    - Pohlig-Hellman Method
  - Non-generic Algorithms: Work only in specific groups, in particular in Z_p
    - The Index Calculus Method
- Remark: Elliptic curves can only be attacked with generic algorithms which are weaker than non-generic algorithms. Hence, elliptic curves are secure with shorter key lengths than the DLP in prime fields Z_p

Attacks against the Discrete Logarithm Problem

Summary of records for computing discrete logarithms in Z_p^*

Decimal digits	Bit length	Date
58	193	1991
68	216	1996
85	282	1998
100	332	1999
120	399	2001
135	448	2006
160	532	2007

In order to prevent attacks that compute the DLP, it is recommended to use primes with a length of at least 1024 bits for schemes such as Diffie-Hellman in Z_p^*

Security of the classical Diffie–Hellman Key Exchange

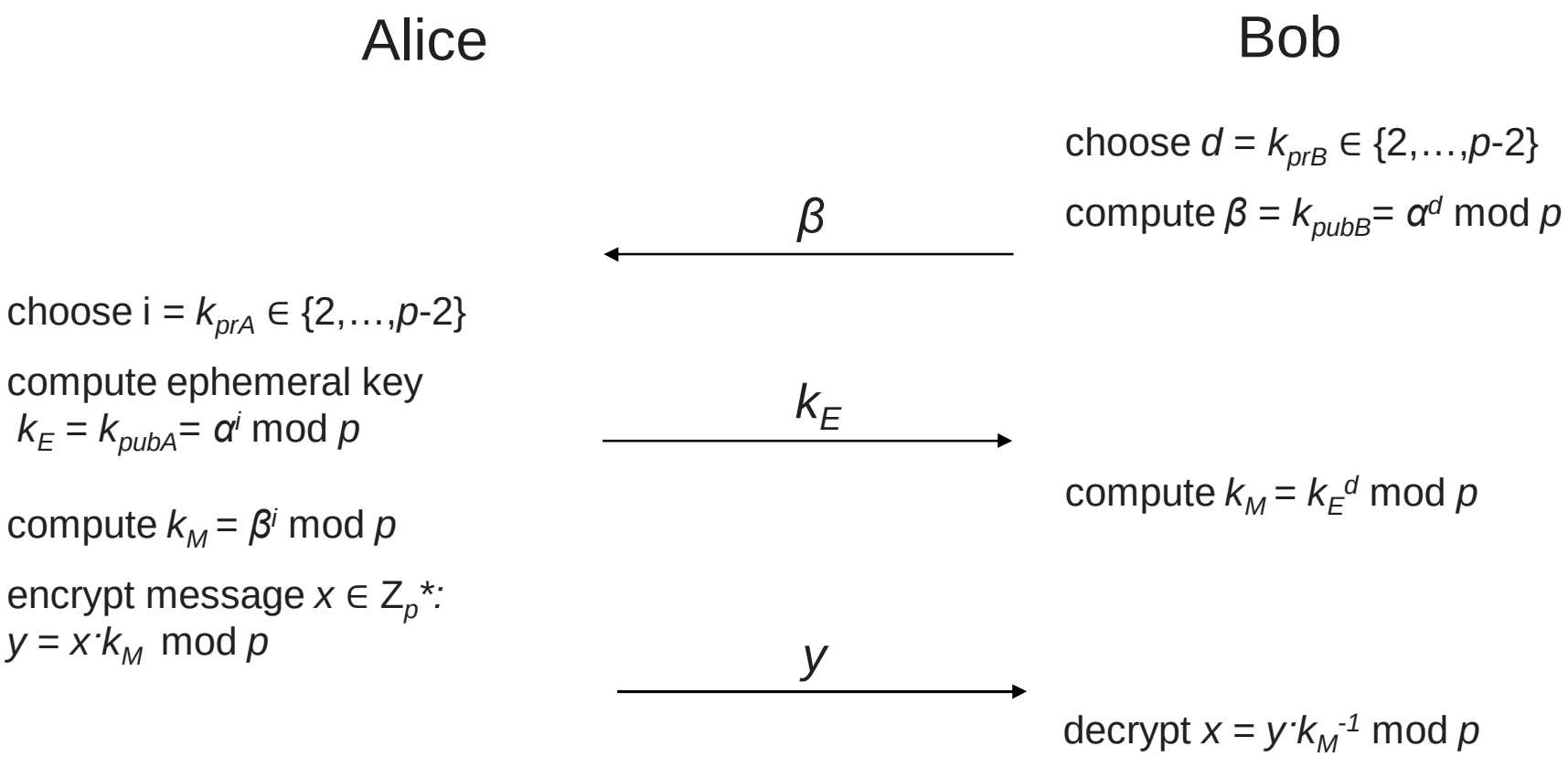
- Which information does Oscar have?
  - α, p
  - $k_{pubA} = A = \alpha^a \bmod p$
  - $k_{pubB} = B = \alpha^b \bmod p$
- Which information does Oscar want to have?
  - $k_{AB} = \alpha^{ba} = \alpha^{ab} = \bmod p$
  - This is kown as Diffie-Hellman Problem (DHP)
- The only known way to solve the DHP is to solve the DLP, i.e.
  1. Compute $a = log_\alpha A \bmod p$
  2. *Compute* $k_{AB} = B^a = \alpha^{ba} = \bmod p$

  It is conjectured that the DHP and the DLP are equivalent, i.e., solving the DHP implies solving the DLP.
- To prevent attacks, i.e., to prevent that the DLP can be solved, choose $p > 2^{1024}$

The Elgamal Encryption Scheme: Overview

- Proposed by Taher Elgamal in 1985
- Can be viewed as an extension of the DHKE protocol
- Based on the intractability of the discrete logarithm problem and the Diffie–Hellman problem

The Elgamal Encryption Scheme: Principle



This looks very similar to the DHKE! The actual Elgamal protocol re-orders the computations which helps to save one communication (cf. next slide)

The Elgamal Encryption Protocol

Alice		Bob
		choose large prime p
		choose primitive element $\alpha \in Z_p^*$ or in a subgroup of Z_p^*
		choose $d = k_{prB} \in \{2,\ldots,p\text{-}2\}$
	$k_{pubB} = (p, \alpha, \beta)$ ←	compute $\beta = k_{pubB} = \alpha^d \bmod p$
choose i = $k_{prA} \in \{2,\ldots,p\text{-}2\}$		
compute $k_E = k_{pubA} = \alpha^i \bmod p$		
compute masking key $k_M = \beta^i \bmod p$		
encrypt message $x \in Z_p^*$: $y = x \cdot k_M \bmod p$	(k_E, y) →	
		compute masking key $k_M = k_E^d \bmod p$
		decrypt $x = y \cdot k_M^{-1} \bmod p$

Computational Aspects

- Key Generation
  - Generation of prime p
  - p has to of size of at least 1024 bits
  - cf. Section 7.6 in *Understanding Cryptography* for prime-finding algorithms
- Encryption
  - Requires two modular exponentiations and a modular multiplictation
  - All operands have a bitlength of $\log_2 p$
  - Efficient execution requires methods such as the square-and-multiply algorithm (cf. Chapter 7)
- Decryption
  - Requires one modular exponentiation and one modulare inversion
  - As shown *in Understanding Cryptography*, the inversion can be computed from the ephemeral key

Security

- Passive attacks
  - Attacker eavesdrops p, α, $\beta = \alpha^d$, $k_E = \alpha^i$, $y = x \cdot \beta^i$ and wants to recover x
  - Problem relies on the DLP

- Active attacks
  - If the public keys are not authentic, an attacker could send an incorrect public key (cf. Chapter 13)
  - An Attack is also possible if the secret exponent i is being used more than once (cf. *Understanding Cryptography* for more details on the attack)

Lessons Learned

- The Diffie–Hellman protocol is a widely used method for key exchange. It is based on cyclic groups.
- The discrete logarithm problem is one of the most important one-way functions in modern asymmetric cryptography. Many public-key algorithms are based on it.
- For the Diffie–Hellman protocol in Z_p^*, *the prime p should be at least 1024 bits* long. This provides a security roughly equivalent to an 80-bit symmetric cipher.
- For a better long-term security, a prime of length 2048 bits should be chosen.
- The Elgamal scheme is an extension of the DHKE where the derived session key is used as a multiplicative masked to encrypt a message.
- Elgamal is a probabilistic encryption scheme, i.e., encrypting two identical messages does not yield two identical ciphertexts.