

# PHY401

## Electromagnetic Theory I

The Frequency Dependence of  
Permittivity

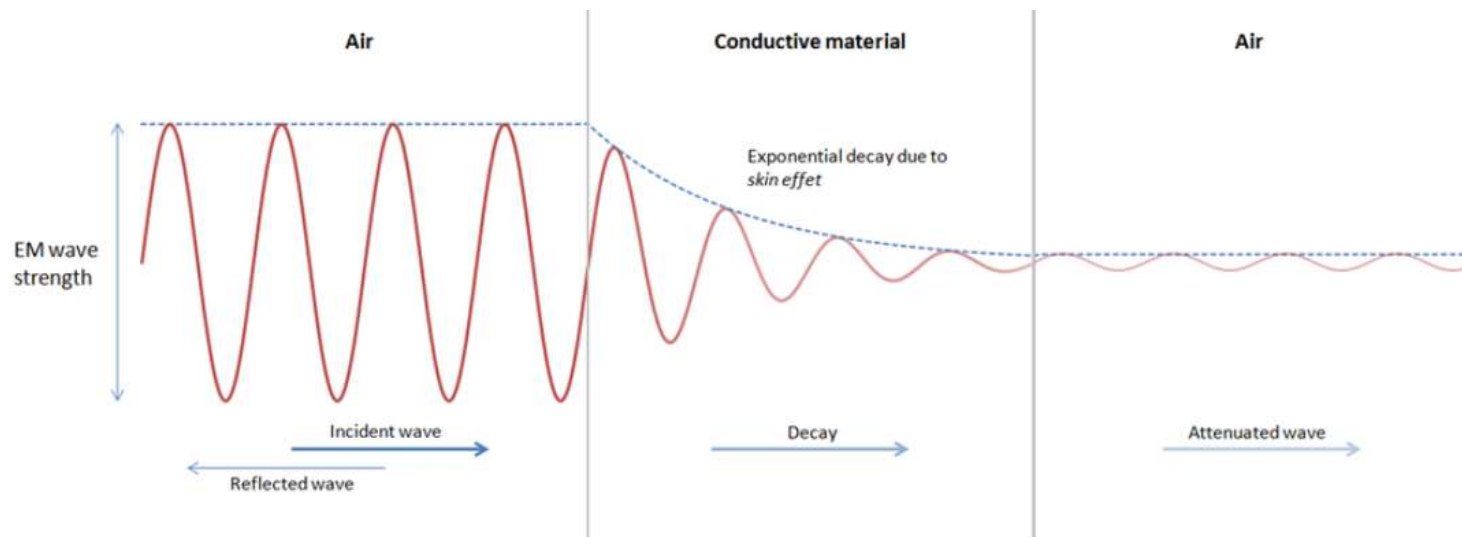
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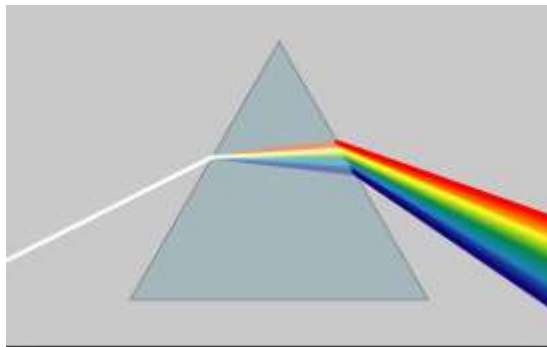


### 9.4.3 The Frequency Dependence of Permittivity

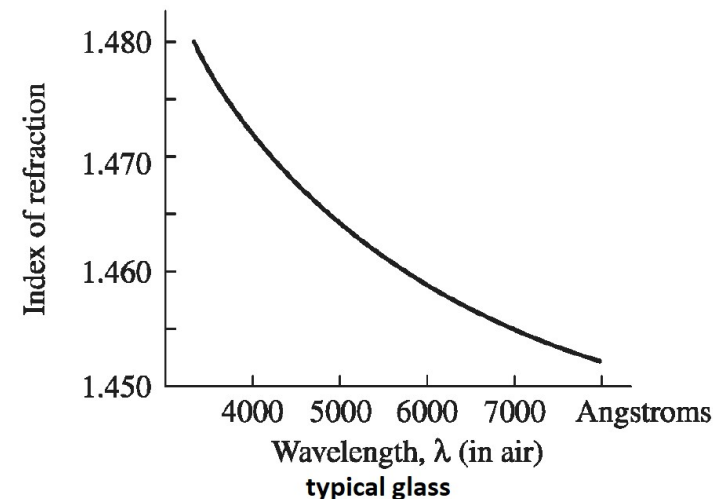
The propagation of electromagnetic waves through matter is governed by three properties of the material: **the permittivity  $\epsilon$** , **the permeability  $\mu$**  and **the conductivity  $\sigma$** .

Each of these parameters depends to some extent **on the frequency** of the waves.

#### Dispersion:



Blue is bent more than red.



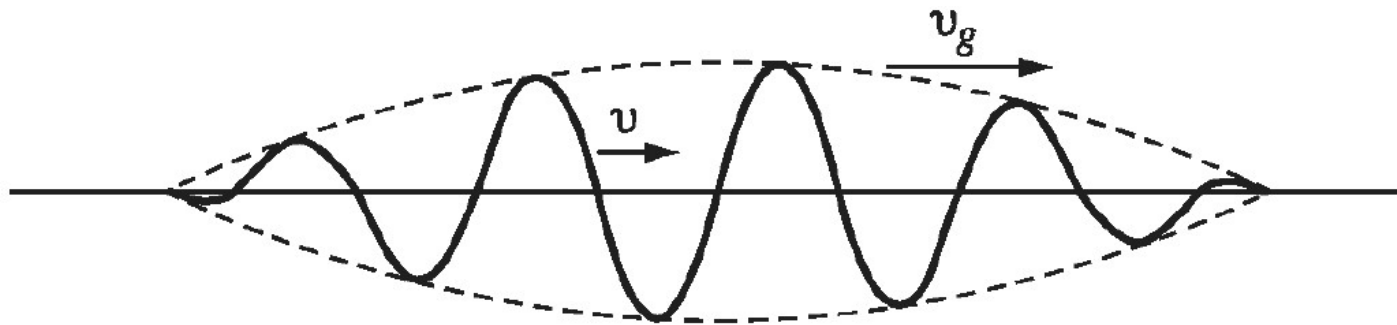
Whenever the speed of a wave depends on its frequency, the supporting medium is called **dispersive**.

Because waves of different frequency travel at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape as it propagates. A sharply peaked wave typically flattens out, whereas each sinusoidal component travels at the ordinary wave (or **phase**) **velocity**,

$$v = \frac{\omega}{k}$$

The packet as a whole (the "envelope") moves at the so-called **group velocity**

$$v_g = \frac{d\omega}{dk}$$



Energy is carried by the group velocity.

In order to understand the nature of the group velocity, let's consider a race. At any given instant, the velocity of a particular runner is different relative to others. Simply, the velocity of each individual runner within the group is different. But, they run within a group. The velocity at which the group of runners is moving can be considered as the *group velocity*.



Group 1



Group 2

## Frequency dependence of permittivity in dielectrics

The electrons in a nonconductor are bound to specific molecules. We shall picture each electron as attached to the end of a spring.

$$F_{binding} = -k_{spring}x = -m\omega_0^2x$$

Any binding force can be approximated this way for sufficiently small displacements from equilibrium, as you can see by expanding the potential energy in a Taylor series about the equilibrium point.

$$U(x) = U(0) + xU'(0) + \frac{1}{2}x^2U''(0) + \dots$$

1st term: Adjust the zero of potential energy so that  $U(0)=0$ .

2nd term:  $dU/dx=-F$  and at zero point  $F(0)=0$

3rd term: The third term is precisely the potential energy of a spring with force constant

$$k_{spring} = \left. \frac{d^2U}{dx^2} \right|_{x=0}$$

Higher terms are neglected.

Meanwhile, there will presumably be some damping force on the electron

$$F_{damping} = -m\gamma \frac{dx}{dt}$$

The driving force of the electron  $F_{driving} = qE = qE_0 \cos(\omega t)$

$E_0$  is the amplitude of the wave at the point  $z$  where the electron is situated

$$m \frac{d^2 x}{dt^2} = F_{binding} + F_{damping} + F_{driving}$$

$$m \frac{d^2 x}{dt^2} + m\gamma \frac{dx}{dt} + m\omega_0^2 x = qE_0 \cos(\omega t)$$

$$\frac{d^2 \tilde{x}}{dt^2} + \gamma \frac{d\tilde{x}}{dt} + \omega_0^2 \tilde{x} = \frac{q}{m} E_0 e^{-i\omega t}$$

In the steady state, the system oscillates at the driving frequency:

$$\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$$

$$\tilde{x}(t)'' = -\omega^2 \tilde{x}_0 e^{-i\omega t} \quad \tilde{x}(t)' = -i\omega \tilde{x}_0 e^{-i\omega t}$$

$$-\omega^2 \tilde{x}_0 - i\omega\gamma \tilde{x}_0 + \omega_0^2 \tilde{x}_0 = \frac{q}{m} E_0$$

$$\tilde{x}_0 (-\omega^2 - i\gamma\omega + \omega_0^2) = \frac{q}{m} E_0$$

$$\tilde{x}_0 = \frac{q/m}{-\omega^2 - i\gamma\omega + \omega_0^2} E_0$$



The resulting dipole moment is the real part of

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2 / m}{\omega_0^2 - \omega^2 - i\gamma\omega} E_0 e^{-i\omega t}$$

In denominator:  $p$  is out of phase with  $E$

In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients. Let's say there are  $f_i$  electrons with frequency  $\omega_j$  and damping  $\gamma_i$  in each molecule. If there are  $N$  molecules per unit volume, the polarization  $P$  is given by the real part of

$$\tilde{P} = \frac{Nq^2}{m} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) \tilde{E}$$

$$\tilde{P} = \epsilon_0 \tilde{\chi}_E \tilde{E}$$

The physical polarization is the real part of complex  $P$  just as the physical field is the real part of complex  $E$ .

$$\tilde{D} = \tilde{\varepsilon} \tilde{E}$$

$$\tilde{P} = \varepsilon_0 \tilde{\chi}_E \tilde{E}$$

$$\tilde{P} = \frac{Nq^2}{m} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right) \tilde{E}$$

$$\tilde{\chi}_E = \frac{Nq^2}{m\varepsilon_0} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

$$\tilde{\varepsilon} = \varepsilon_0 (1 + \tilde{\chi}_E)$$

Relative permittivity is:

$$\tilde{\varepsilon}_r = \frac{\tilde{\varepsilon}}{\varepsilon_0} = 1 + \tilde{\chi}_E = 1 + \frac{Nq^2}{m\varepsilon_0} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j \omega} \right)$$

Ordinarily, the imaginary term is negligible; however, when  $\omega$  is very close to one of the resonant frequencies,  $\omega_j$ , it plays an important role.

In a dispersive medium, the wave number is complex.

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_0}\omega$$

$$\tilde{k} = k + i\kappa$$

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz - \omega t)}$$

The wave is *attenuated* (this is hardly surprising, since the damping absorbs energy).

The absorption coefficient is  $\alpha \equiv 2\kappa$ , since intensity is proportional to  $E^2$ .

Wave velocity,  $\mathfrak{v}$  is  $\omega/k$  and the index of refraction is,  $n=c/\mathfrak{v}=ck/\omega$

$$\tilde{k} = \frac{\omega}{c} \sqrt{\tilde{\epsilon}_r}$$

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \tilde{\chi}_E = 1 + \frac{Nq^2}{m\epsilon_0} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right)$$

For gases, the second term is small, and we can approximate the square root by the first term in the binomial expansion:

$$\sqrt{\tilde{\epsilon}_r} = \sqrt{1 + \tilde{\chi}_E} = 1 + \frac{1}{2} \tilde{\chi}_E$$

$$\tilde{k} = \frac{\omega}{c} \sqrt{\tilde{\epsilon}_r} = \frac{\omega}{c} \left[ 1 + \frac{Nq^2}{2m\epsilon_0} \left( \sum_j \frac{q^2 / m}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right) \right]$$

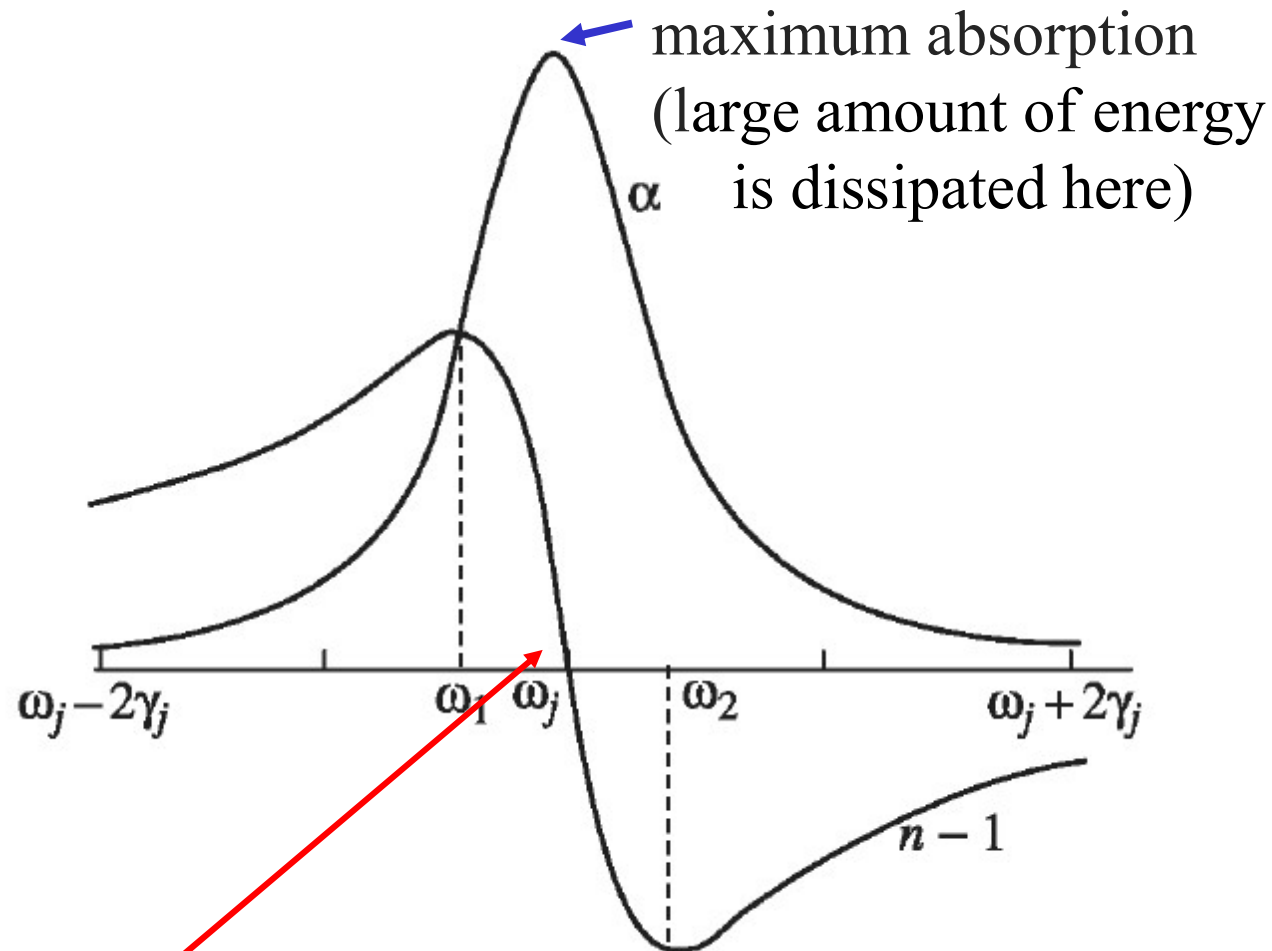
$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0} \left( \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2} \right)$$

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

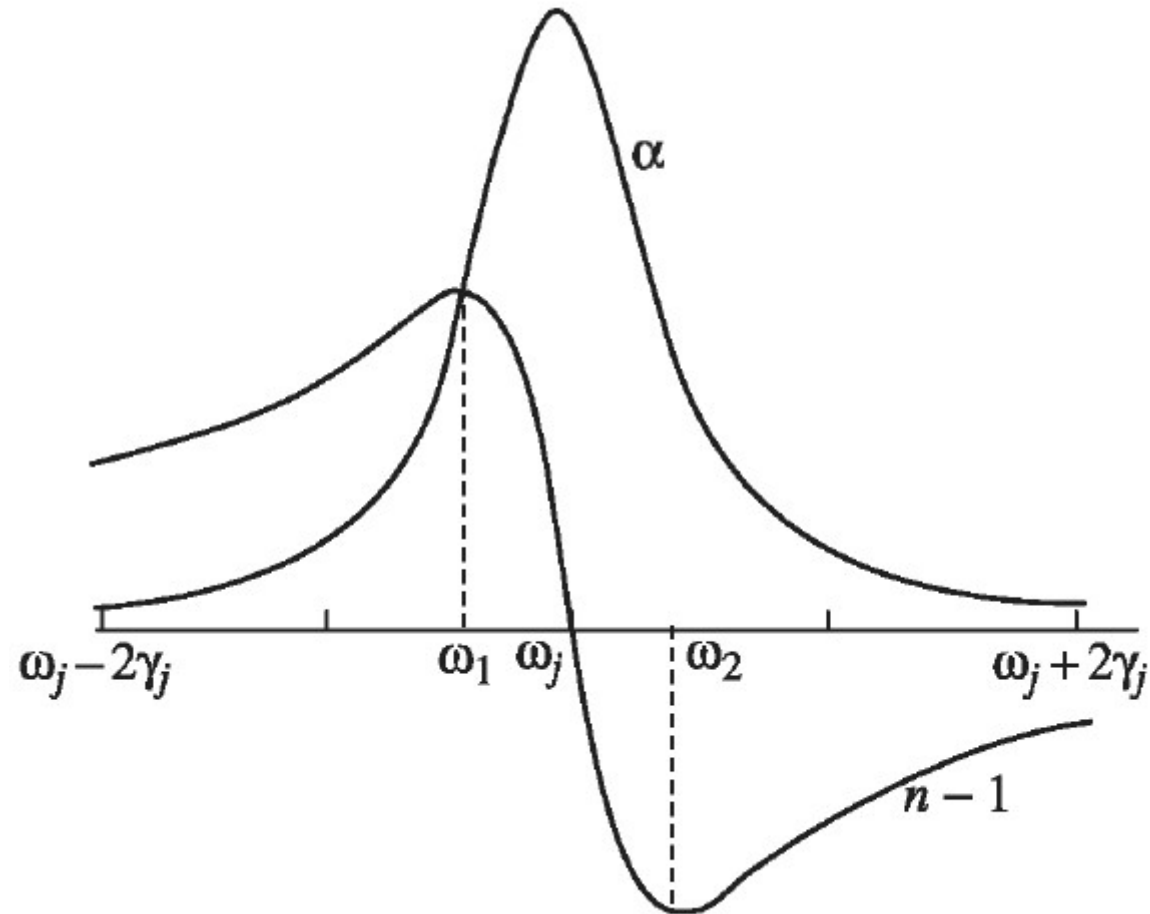
$n$  is ordinarily a slowly increasing function of  $\omega$ .

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The index of refraction and absorption in the vicinity of one of the resonances



In the immediate neighborhood of a resonance (between  $\omega_1$  and  $\omega_2$ ), the index of refraction drops sharply -> called **anomalous dispersion**



$n$  runs below 1 above the resonance, suggesting that the wave speed exceeds  $c$ . Energy does not travel at the wave velocity, it travels at group velocity. **The phase velocity can exceed  $c$  in the neighborhood of a resonance.** (see Prob. 9.26)

If you stay away from the resonances, the damping can be ignored ( $\gamma_j \approx 0$ ) and the formula for the index of refraction simplifies:

$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_0} \left( \sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \right) \longrightarrow n = 1 + \frac{Nq^2}{2m\epsilon_0} \left( \sum_j \frac{f_j}{\omega_j^2 - \omega^2} \right)$$

For transparent materials, the nearest significant resonances typically lie in the ultraviolet, so that  $\omega < \omega_j$ . In that case,

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2} \left( 1 - \frac{\omega^2}{\omega_j^2} \right)^{-1} \cong \frac{1}{\omega_j^2} \left( 1 + \frac{\omega^2}{\omega_j^2} \right)$$

$$n = 1 + \frac{Nq^2}{2m\epsilon_0} \left( \sum_j f_j \frac{1}{\omega_j^2} \left( 1 + \frac{\omega^2}{\omega_j^2} \right) \right)$$

$$n = 1 + \left( \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \right) + \omega^2 \left( \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \right)$$

$$n = 1 + \left( \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \right) + \omega^2 \left( \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \right)$$

In terms of wavelength in vacuum which is  $\lambda = \frac{2\pi c}{\omega}$

$$A = \left( \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \right)$$

$$n = 1 + A + \frac{AB}{\lambda^2} \longrightarrow \boxed{n = 1 + A \left( 1 + \frac{B}{\lambda^2} \right)}$$

This is known as Cauchy's formula; the constant  $A$  is called the coefficient of refraction, and  $B$  is called the coefficient of dispersion. Cauchy's equation applies reasonably well to most gases, in the optical region.

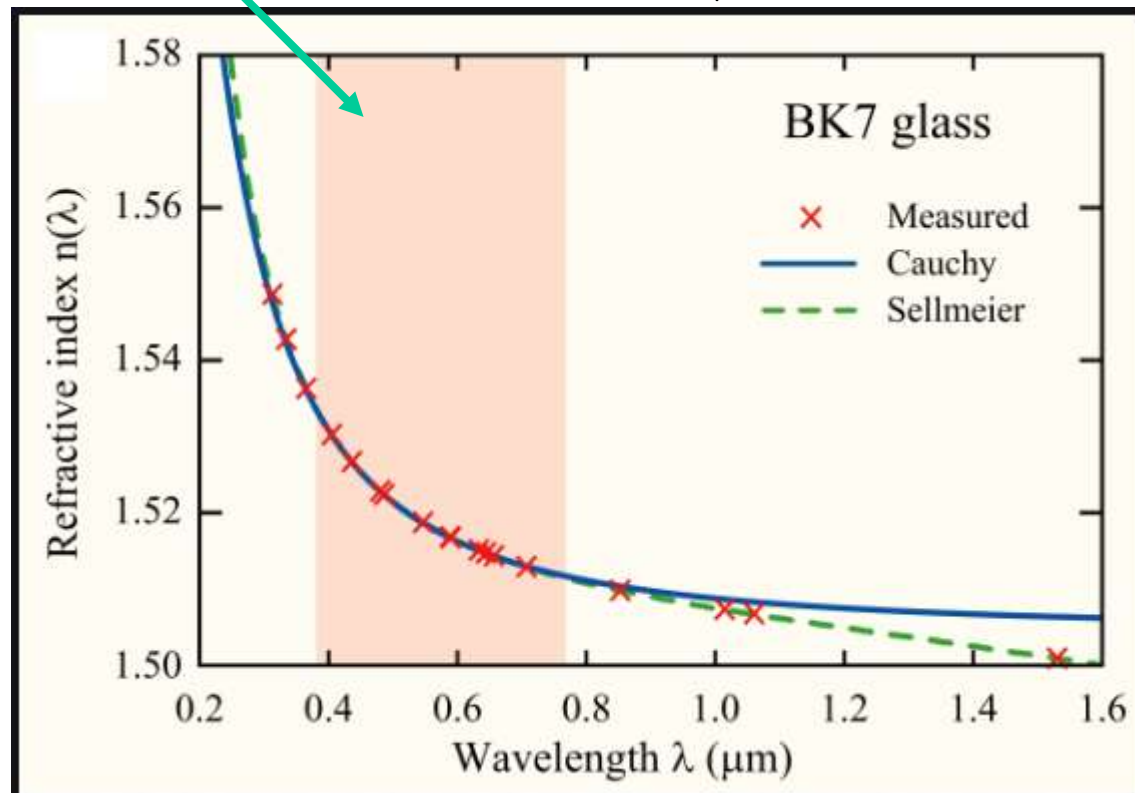


**Cauchy's equation** is an empirical relationship between the refractive index and wavelength of light for a particular transparent material. It is named for the mathematician Augustin-Louis **Cauchy**, who defined it in 1836.

The most general form of Cauchy's equation is

$$n(\lambda) = B + \frac{C}{\lambda^2} + \frac{D}{\lambda^4} + \dots$$

Visible  
region



**Cauchy's equation** is only valid for regions of normal dispersion in the visible wavelength region. In the infrared, the equation becomes inaccurate, and **it cannot represent regions of anomalous dispersion**. Despite this, its mathematical simplicity makes it useful in some applications.

The Sellmeier equation is a later development of Cauchy's work that handles anomalously dispersive region:

The usual form of the equation for glasses is

$$n^2(\lambda) = 1 + \frac{B_1\lambda^2}{\lambda^2 - C_1} + \frac{B_2\lambda^2}{\lambda^2 - C_2} + \frac{B_3\lambda^2}{\lambda^2 - C_3},$$

where  $n$  is the refractive index,  $\lambda$  is the wavelength, and  $B_{1,2,3}$  and  $C_{1,2,3}$  are experimentally determined *Sellmeier coefficients*. These coefficients are usually quoted for  $\lambda$  in micrometres.

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