PHY401
Electromagnetic Theory I

Guided Waves in
Rectangular Waveguides
and Coaxial Transmission
Lines

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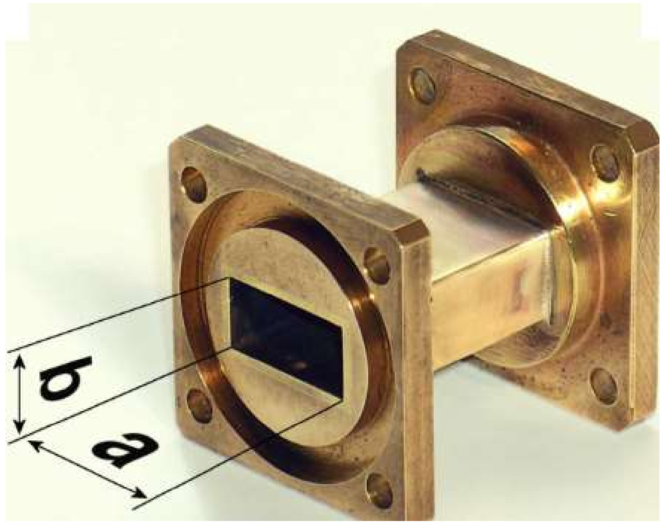
9.5.1 Waveguides

Now lets deal with electromagnetic waves confined to the interior of a hollow pipe, or waveguide (not infinite extent in xy).

Waveguides generally made of good conductor, so that **E**=0 and **B**=0 inside the material.



The boundary conditions at the inner wall are: $E_{//} = 0 \; and \; B_{\perp} = 0$

The generic form of the monochromatic waves:

$$\tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(\tilde{k}z-\omega t)}$$

$$\tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(\tilde{k}z-\omega t)}$$

In the interior of the waveguide, the waves satisfy Maxwell's equations:

$$(i)\nabla.\vec{E}=0 \qquad (ii)\nabla.\vec{B}=0$$

$$(iii)\nabla\times\vec{E}=-\frac{\partial B}{\partial t} \qquad (iv)\nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Notice that we took $\rho_f=0 \text{ and } J_f=0$

Lets find $\tilde{E}_0$ *and* $\tilde{B}_0$

Confined waves are *not* (in general) transverse; in order to fit the boundary conditions we shall have to include longitudinal components (*Ez* and *B*z).

$$\tilde{E}_0=E_x\hat{x}+E_y\hat{y}+E_z\hat{z} \qquad \tilde{B}_0=B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$$

$$(iii)\nabla \times \vec{E} = -\frac{\partial B}{\partial t} \qquad (iv)\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\widetilde{E}_0 = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} \qquad \widetilde{B}_0 = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$(i)\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \qquad (iv)\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$

$$(ii)\frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x \qquad (v)\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$$

$$(iii)ikE_x - \frac{\partial E_z}{\partial x} - = i\omega B_y \qquad (vi)ikB_x - \frac{\partial B_z}{\partial x} - = -\frac{i\omega}{c^2}E_y$$

Equations (ii), (iii), (v), and (vi) can be solved for E_x, E_y, B_x, and B_y. For example for E_x, use (iii) and (v).

We'll wait for any accumulated free charge to disappear. From then on, $\rho_f = 0$, and we have

$$\text{(i)} \quad E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right),$$

$$\text{(ii)} \quad E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right),$$

$$\text{(iii)} \quad B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right),$$

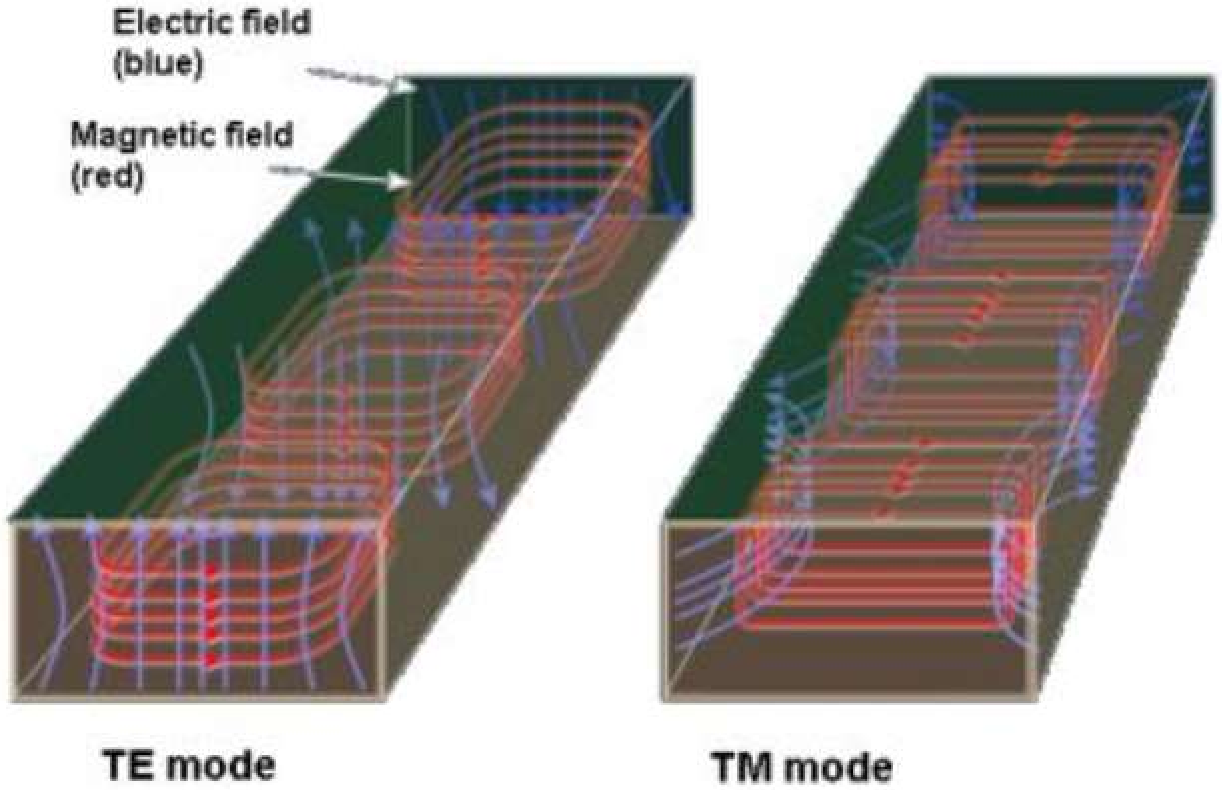
$$\text{(iv)} \quad B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).$$

It suffices, then, to determine the longitudinal components *Ez* and *Bz*. Inserting these equations into the Maxwell's equations (i) and (ii), it yields uncoupled equations for E_z and B_z.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{v^2}-k^2\right]E_z=0$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\omega^2}{v^2}-k^2\right]B_z=0$$

If $E_z=0 \Rightarrow$ TE (transverse electric) waves;
If $B_z=0 \Rightarrow$ TM (transverse magnetic) waves;
If $E_z=0$ and $B_z=0 \Rightarrow$ TEM waves.



Hollow, metallic waveguides are compatible with TE and TM modes only. This is in contrast to coaxial cable where TEM mode can propagate.

Question: Prove that TEM waves cannot occur in a hollow wave guide.

If $Ez = 0$, Gauss's law says,

$$(i)\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=0$$

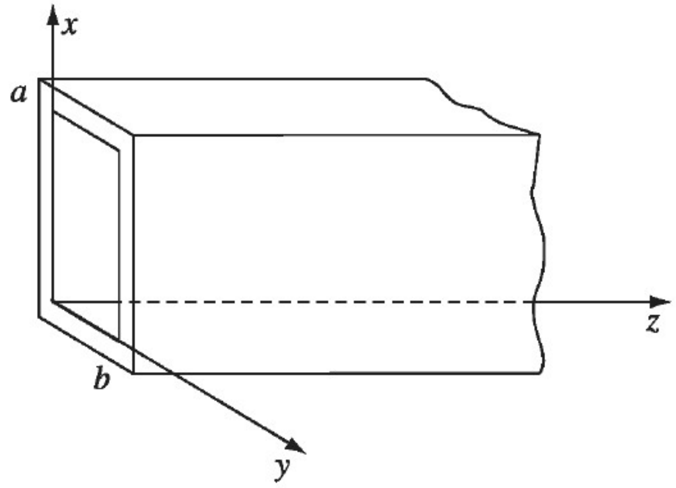
If $Bz = 0$, Faraday's law says,

$$(ii)\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=0$$

In magnetostatics we have learned that magnetic flux lines always close upon themselves. Magnetic fields are close loops and from Amper law we know that if we have a magnetic field, there should be a current in between which produces it. However, as there is no current in the middle of waveguide, TEM cannot be supported in such a one conductor structure. If we have a hollow waveguide with a center conductor such as in a coaxial cable, however, TEM waves are supported.

9.5.2 TE Waves in a Rectangular Waveguide

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+(\omega/c)^2-k^2\right]B_z=0$$



$$E_z=0 \longrightarrow B_z(x,y)=X(x)\,\mathrm{Y}(\mathrm{y})$$

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)=0$$

$$\frac{1}{X}\frac{\partial^2}{\partial x^2}=-k_x^{\ 2} \qquad \frac{1}{Y}\frac{\partial^2}{\partial y^2}=-k_y^{\ 2}$$

$$\frac{\omega^2}{c^2}=k^2+k_x^{\ 2}+k_y^{\ 2}$$

$$X(x)=A\sin k_x x+B\cos k_x x \qquad Y(y)=C\sin k_y y+D\cos k_y y$$

$$\left.\frac{dX}{dx}\right|_{x=0}=A\cos k_x x-B\sin k_x x=0$$

$$A=0$$

$$\left.\frac{dY}{dy}\right|_{y=0}=C\cos k_y y-D\sin k_y y=0$$

$$C=0$$

$$\left.\frac{dX}{dx}\right|_{x=a}=-B\sin k_x a=0$$

$$\left.\frac{dY}{dy}\right|_{y=b}=-D\sin k_y b=0$$

$$k_x=m\pi/a,\ m=0,1,2,\ldots \qquad k_y=n\pi/b,\ n=0,1,2,\ldots$$

$$X(x)=B\cos(m\pi x/a) \qquad Y(y)=D\cos(n\pi y/b)$$

We conclude that the TE_{mn} mode is:

$$B_z(x,y) = B_0 \cos(m\pi x/a)\cos(n\pi x/b)$$

The first index is conventionally associated with the *larger* dimension, so we assume *a*>*b*.

The wave number is: $\dfrac{\omega^2}{c^2} = k^2 + k_x{}^2 + k_y{}^2$

$$k = \sqrt{(\omega/c)^2 - \pi^2\left[(m/a)^2 + (n/b)^2\right]}$$

The wave number is imaginary if $(\omega/c)^2 < \pi^2\left[(m/a)^2 + (n/b)^2\right]$

$$\omega < c\pi\sqrt{\left[(m/a)^2 + (n/b)^2\right]} = \omega_{mn}$$

$$\omega_{mn} = c\pi\sqrt{\left[(m/a)^2 + (n/b)^2\right]}$$

When *k* is imaginary, instead of a traveling wave we have exponentially attenuated fields. For this reason, ω_{mn} is called the **cut-off frequency.**

The *lowest* cutoff frequency for a given waveguide occurs for the mode TE_{10}

$$\omega_{mn} = c\pi\sqrt{\left[(m/a)^2 + (n/b)^2\right]}$$

$$\omega_{10} = c\pi\sqrt{\left[(1/a)^2\right]} = c\pi / a$$

Frequencies less than this will not propagate at all.

The wave number can be written more simply in terms of the cutoff frequency:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}{}^2}$$

The wave velocity is

$$\vartheta = \frac{\omega}{k} = \frac{c}{\sqrt{1-\left(\omega_{mn}/\omega\right)^2}}$$

which is greater than *c* at and below cut-off frequency.

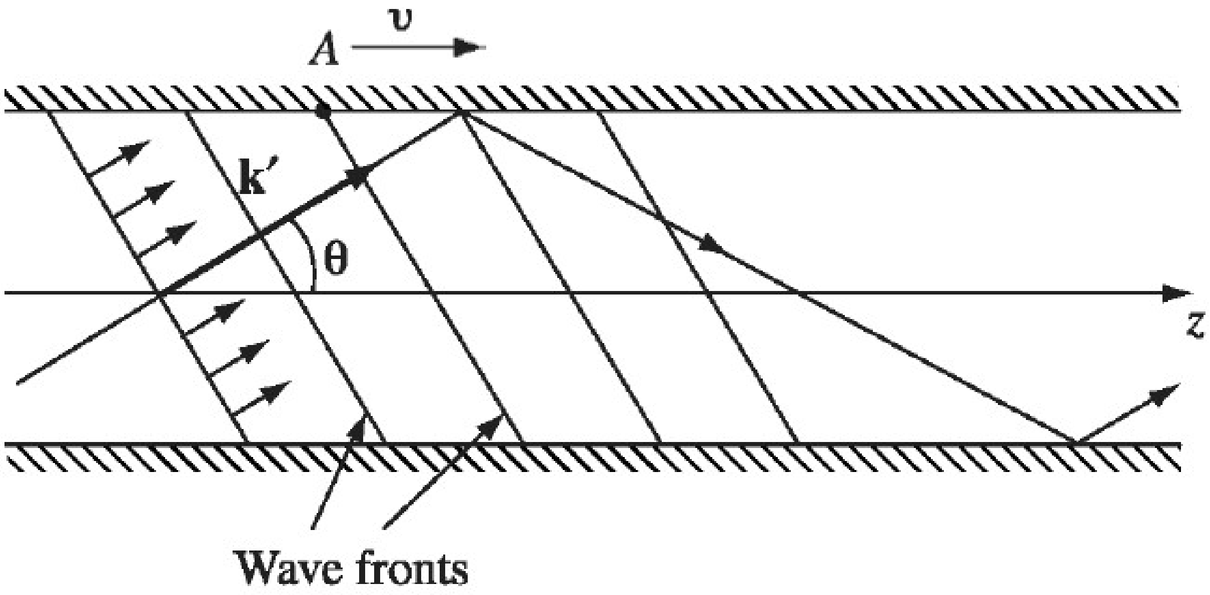
However, the energy carried by the wave travels at the *group* velocity:

$$\vartheta_g = \frac{1}{dk/d\omega} = c\sqrt{1-\left(\omega_{mn}/\omega\right)^2} < c$$

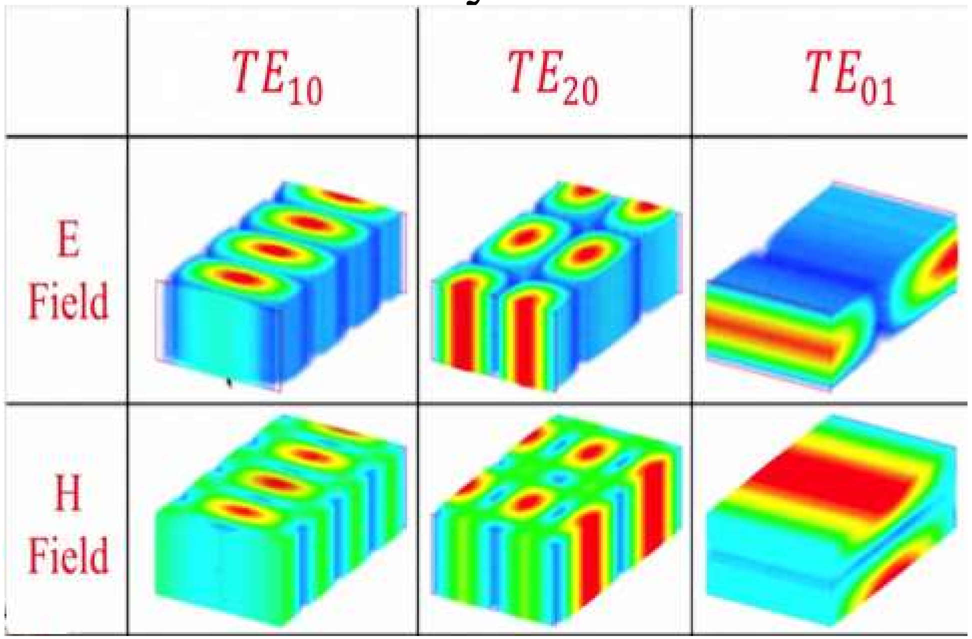
Consider an ordinary plane wave, traveling at an angle θ to the z axis, reflecting perfectly off each conducting surface. In the x and y directions, the (multiply reflected) waves interfere to form standing wave patterns.

$$\mathbf{k}' = \frac{\pi m}{a}\hat{\mathbf{x}} + \frac{\pi n}{b}\hat{\mathbf{y}} + k\hat{\mathbf{z}}$$

$$\omega = c|\mathbf{k}'| = c\sqrt{k^2 + \pi^2[(m/a)^2 + (n/b)^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}$$



Now, let us see how fields vary in TE mode.



-There is half sinusoidal variation along *x* direction and along *y* direction there is no field variation. So, this will be TE_{10} mode.

-In the *x* direction, there are 2 half sinusoidal variations of the field and in the *y* direction there is no variation of the field. So, this is TE_{20} mode.

-In *x* direction there is no variation of the field and in *y* direction, there is one half sinusoidal variation of the field. So, this is TE_{01} mode.

Reference:Prof. Girish Kumar’s lecture notes

Rectangular waveguide

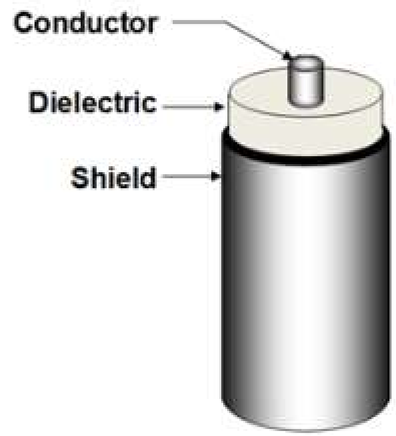
Waveguide to coax adapter

Waveguide bends

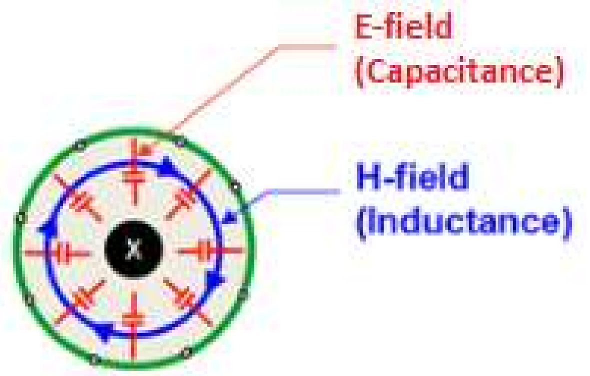
E-tee

9.5.3 The Coaxial Transmission Line

A coaxial transmission line admits modes with $E_z = 0$ and $B_z = 0$.



From 2D Maxwell's equations



$$k = w / c$$

$$cB_y = E_x \ \ and \ \ cB_x = -E_y$$

Electrostatics (electric charges at rest)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0, \ \ and \ \ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

Magnetostatics (steady currents)

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0, \ \ and \ \ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

The solution with cylindrical symmetry can be borrowed directly from the case of an infinite line charge and an infinite straight current:

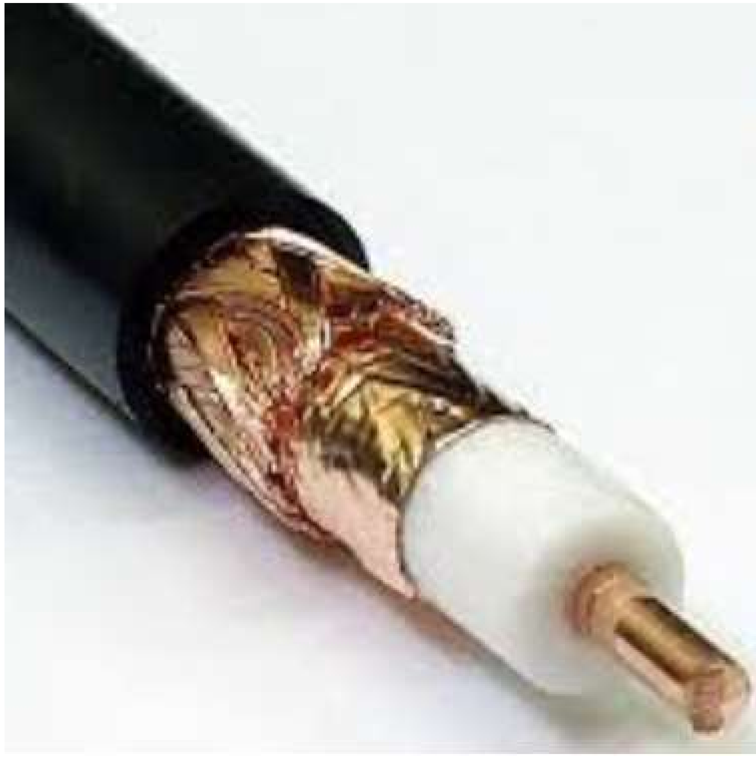
$$\mathbf{E}_0(s,\phi) = \frac{A}{s}\hat{\mathbf{s}}, \quad \mathbf{B}_0(s,\phi) = \frac{A}{cs}\hat{\boldsymbol{\varphi}}$$

Taking the real part:

$$\mathbf{E}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{s}\hat{\mathbf{s}}$$

$$\mathbf{B}(s,\phi,z,t) = \frac{A\cos(kz-\omega t)}{cs}\hat{\boldsymbol{\varphi}}$$

Coaxial cables are also used in automobiles, aircraft, military and medical equipment, as well as to connect satellite dishes, radio and television antenna to their respective receivers.



The coaxial transmission lines

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