PHY401
Electromagnetic Theory I

Retarded Potentials and
Jefimenko's Equations

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10.2.1 Retarded Potentials

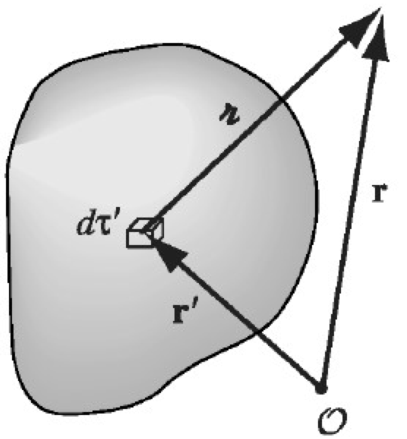
$$\nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho \qquad \xrightarrow{\text{static case}} \qquad \nabla^2 V = -\frac{1}{\varepsilon_0}\rho$$

$$\nabla^2 \mathbf{A} - \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad\qquad \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

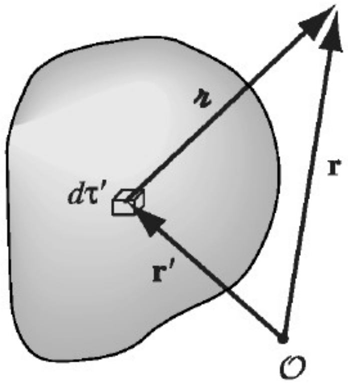
With the familiar solutions,

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{\text{r}} d\tau'$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\text{r}} d\tau' \qquad \text{Eq.(10.24)}$$



In the nonstatic case, it is not the status of the source right now that matters, but rather its condition at some earlier time t_r when the "message" left.



The retarded time:

$$t_r = t - \frac{\mathscr{r}}{c}$$

The natural generalization of Eq. 10.24 for nonstatic sources is therefore: Because the integrands are evaluated at the retarded time, these are called **retarded potentials.**

$$V(\mathscr{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\boldsymbol{r}', t_r)}{\mathscr{r}} d\tau'$$

$$\boldsymbol{A}(\mathscr{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\boldsymbol{r}', t_r)}{\mathscr{r}} d\tau'$$

Argument: The light we see now left each star at the retarded time corresponding to that star's distance from the earth.

Let’s prove that these equations satisfy the inhomogeneous wave equation.

Retarded Scalar Potential Satisfies the Lorenz Gauge Condition

$$V(\boldsymbol{\mathscr{r}}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r', t_r)}{\boldsymbol{\mathscr{r}}}\, d\tau' \qquad\qquad \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{1}{\varepsilon_0}\rho$$

Proof:

$$\nabla V = \frac{1}{4\pi\varepsilon_0}\int \nabla\left(\frac{\rho(r', t_r)}{\boldsymbol{\mathscr{r}}}\right) d\tau' = \frac{1}{4\pi\varepsilon_0}\int \frac{\boldsymbol{\mathscr{r}}(\nabla\rho) - \rho(\nabla\boldsymbol{\mathscr{r}})}{\boldsymbol{\mathscr{r}}^2}\, d\tau'$$

Using quotient rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

$$\nabla\rho = \nabla\rho(r', t_r) = \frac{\partial\rho}{\partial t_r}\nabla t_r = \dot{\rho}\left(-\frac{1}{c}\right)\nabla\boldsymbol{\mathscr{r}} \qquad\qquad \nabla\boldsymbol{\mathscr{r}} = \hat{\boldsymbol{\mathscr{r}}}$$

$$\nabla V = -\frac{1}{4\pi\varepsilon_0}\int\left(\frac{\dot{\rho}\hat{r}}{c\boldsymbol{\mathscr{r}}} + \frac{\rho\hat{r}}{\boldsymbol{\mathscr{r}}^2}\right) d\tau'$$

$$\nabla . \nabla V = \nabla^2 \mathrm{V} = -\frac{1}{4\pi\varepsilon_0} \int \nabla . \left(\frac{\dot{\rho}\hat{\boldsymbol{\mathscr{r}}}}{c\boldsymbol{\mathscr{r}}} + \frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right) d\tau'$$

$$\nabla . \left[\frac{\dot{\rho}\hat{\boldsymbol{\mathscr{r}}}}{c\boldsymbol{\mathscr{r}}} + \frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right] = \frac{1}{c} \nabla . \left(\frac{\dot{\rho}\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}} \right) + \nabla . \left(\frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right)$$

$$= \frac{1}{c} \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}} \nabla \dot{\rho} + \dot{\rho} \nabla \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}} \right] + \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \nabla \rho + \rho \nabla \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right]$$

$$\nabla \dot{\rho} = \nabla \dot{\rho}(r', t_r) = \frac{\partial \dot{\rho}}{\partial t_r} \nabla t_r = \ddot{\rho} \frac{-1}{c} \nabla \boldsymbol{\mathscr{r}} = -\frac{\ddot{\rho}}{c} \hat{\boldsymbol{\mathscr{r}}} \quad and \quad \nabla \rho = \frac{-\dot{\rho}}{c} \hat{\boldsymbol{\mathscr{r}}}$$

$$\nabla \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}} = \frac{1}{\boldsymbol{\mathscr{r}}^2} \quad and \quad \nabla \frac{\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} = 4\pi \delta^3(\boldsymbol{\mathscr{r}}) \qquad \nabla^2 \mathrm{V} - \mu_0 \varepsilon_0 \frac{\partial^2 V}{\partial \mathrm{t}^2} = -\frac{1}{\varepsilon_0} \rho$$

$$\nabla . \left[\frac{\dot{\rho}\hat{\boldsymbol{\mathscr{r}}}}{c\boldsymbol{\mathscr{r}}} + \frac{\rho\hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2} \right] = \frac{1}{c} \left[-\frac{\ddot{\rho}}{c\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}}{\boldsymbol{\mathscr{r}}^2} \right] + \left[-\frac{1}{\boldsymbol{\mathscr{r}}^2} \frac{\dot{\rho}}{c} + 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}}) \right]$$

$$= -\frac{1}{c^2} \ddot{\rho} + 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla^2 V = -\frac{1}{4\pi\varepsilon_0}\int\left[-\frac{1}{c^2}\ddot{\rho} + 4\pi\rho\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau'$$

$$= \frac{1}{c^2}\int\frac{\ddot{\rho}}{4\pi\varepsilon_0\boldsymbol{\mathscr{r}}}d\tau' - \frac{\rho(r,t)}{\varepsilon_0}$$

$$\int\frac{\ddot{\rho}}{4\pi\varepsilon_0\boldsymbol{\mathscr{r}}}d\tau' = \int\frac{1}{4\pi\varepsilon_0\boldsymbol{\mathscr{r}}}\frac{\partial^2\rho}{\partial {t_r}^2}d\tau' = \frac{\partial^2}{\partial {t_r}^2}\int\frac{\rho}{4\pi\varepsilon_0\boldsymbol{\mathscr{r}}}d\tau' = \frac{\partial^2 V}{\partial {t_r}^2} = \frac{\partial^2 V}{\partial t^2}$$

$$\nabla^2 V = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{\rho(\boldsymbol{\mathscr{r}},t)}{\varepsilon_0}$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho(\boldsymbol{\mathscr{r}},t)}{\varepsilon_0}$$

The retarded scalar potential satisfies the inhomogeneous wave equation under Lorenz gauge condition.

Retarded Vector Potential Satisfies the Lorenz Gauge Condition

Show that the retarded vector potential satisfies the Lorenz gauge condition.

$$A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t_r)}{\mathcal{r}}\,d\tau' \qquad\qquad \nabla^2 A - \mu_0\varepsilon_0\frac{\partial^2 A}{\partial t^2} = -\mu_0 J$$

Solution:

$$\nabla.\left(\frac{J(r',t_r)}{\mathcal{r}}\right) = \frac{\mathcal{r}(\nabla.J) - J(\nabla\mathcal{r})}{\mathcal{r}^2} \qquad\qquad t_r \equiv t - \frac{|\mathcal{r} - \mathcal{r}'|}{c}$$

Using quotient rule: $\nabla\left(\frac{\boldsymbol{A}}{g}\right) = \frac{g\nabla.\boldsymbol{A} - \boldsymbol{A}(\nabla g)}{g^2}$

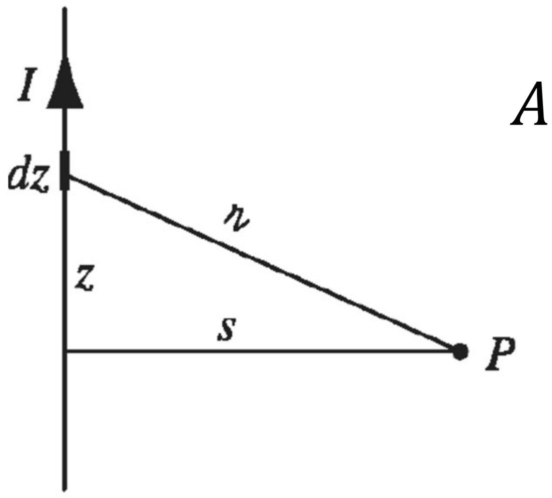
Solve Problem 10.8.

Example 10.2 An infinite straight wire carries the current

$$I(t) = \begin{cases} 0 & \text{for } t \le 0 \\ I_0 & \text{for } t > 0 \end{cases}$$

Find the resulting electric and magnetic fields.

Solution: The wire is electrically neutral, so the scalar potential is zero.



$$A(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{\infty} \frac{I(t_r)}{\mathscr{r}}\,dz$$

For $t < s/c$, the "news" has not yet reached P, and the potential is zero.
For $t > s/c$, only the segment $|z| \le \sqrt{(ct)^2 - s^2}$ contributes.

$$A(s,t) = \left(\frac{\mu_0 I_0}{4\pi}\hat{z}\right) 2 \int_0^{\sqrt{(ct)^2 - s^2}} \frac{dz}{\sqrt{s^2 + z^2}}$$

$$= \frac{\mu_0 I_0}{2\pi}\hat{\mathbf{z}} \ln\left(\sqrt{s^2 + z^2} + z\right)\Big|_0^{\sqrt{(ct)^2 - s^2}} = \frac{\mu_0 I_0}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)\hat{\mathbf{z}}$$

The electric field is $\vec{E}(s,t) = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\hat{z}$

The magnetic field is $\vec{B}(s,t) = \vec{\nabla} \times \vec{A} = -\frac{\partial \vec{A}_z}{\partial s}\hat{\phi} = \frac{\mu_0 I_0}{2\pi s} \frac{ct}{\sqrt{(ct)^2 - s^2}}\hat{\phi}$

Notice that as $t \to \infty$ we recover the static case: $E = 0$, $\vec{B} = \frac{\mu_0 I_0}{2\pi s}\hat{\phi}$

10.2.2 Jefimenko's Equations

Given the retarded potentials,

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r',t_r)}{\mathscr{r}}\,d\tau' \qquad A(r,t) = \frac{\mu_0}{4\pi}\int \frac{J(r',t_r)}{\mathscr{r}}\,d\tau'$$

in principle, the fields can be determined: $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \vec{B} = \vec{\nabla}\times\vec{A}$

The integrands depend on r both explicitly, through $\mathscr{r} = |r - r'|$ in the denominator, and implicitly, through the retarded time $t_r = t - \mathscr{r}/c$ in the argument of the numerator.

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]d\tau' \quad \text{(as calculated before)}$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\vec{J}}}{\mathscr{r}}\,d\tau'$$

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Putting them together (and using $c^2=1/(\mu_0\varepsilon_0)$):

$$\boxed{\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}+\frac{\dot{\rho}(\mathbf{r}',t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}}-\frac{\dot{\mathbf{J}}(\mathbf{r}',t_r)}{c^2\mathscr{r}}\right]d\tau'.}$$

This is the time-dependent generalization of Coulomb's law.
In the static case, the second and third terms drop out and the first term loses its dependence on t_r.

As for **B**, the curl of **A** contains two terms:

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\int\nabla\times\frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathrm{r}}d\tau'=\frac{\mu_0}{4\pi}\int[\frac{1}{\mathrm{r}}\nabla\times\mathbf{J}-\mathbf{J}\times\nabla\frac{1}{\mathrm{r}}]d\tau'$$

$$(\nabla\times\mathbf{J})_x=\frac{\partial J_z}{\partial y}-\frac{\partial J_y}{\partial z}\qquad(\nabla\times\mathbf{J})_x=-\frac{1}{c}\left(\dot{J}_z\frac{\partial\mathscr{r}}{\partial y}-\dot{J}_y\frac{\partial\mathscr{r}}{\partial z}\right)=\frac{1}{c}\left[\dot{\mathbf{J}}\times(\nabla\mathscr{r})\right]_x$$

$$\frac{\partial J_z}{\partial y}=\dot{J}_z\frac{\partial t_r}{\partial y}=-\frac{1}{c}\dot{J}_z\frac{\partial\mathscr{r}}{\partial y}\qquad\frac{\partial J_y}{\partial z}=\dot{J}_y\frac{\partial t_r}{\partial z}=-\frac{1}{c}\dot{J}_y\frac{\partial\mathscr{r}}{\partial z}$$

$\nabla \mathscr{r} = \hat{\mathscr{r}}$ so $\nabla \times \mathbf{J} = \frac{1}{c}\dot{\mathbf{J}} \times \hat{\mathscr{r}}$

Meanwhile $\nabla(1/\mathscr{r}) = -\hat{\mathscr{r}}/\mathscr{r}^2$, and hence

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}} \right] \times \hat{\mathscr{r}}\, d\tau'}$$

This is the time-dependent generalization of the Biot-Savart law.

The earliest explicit statement of $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ solutions to Maxwell's equations was in 1966 by Oleg Jefimenko.

In practice **Jefimenko's equations** are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory.

To get to the *retarded potentials,* all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of the *fields* completely new terms (involving derivatives of ρ and **J**) appear.