PHY401
Electromagnetic Theory I

Lienard-Wiechert Potentials and the Fields of a Moving Charge

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Contents

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10.3.1 Lienard-Wiechert Potentials

Let us calculate the retarded potentials for a point charge that is moving on a specified trajectory.

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t$$

The retarded time is t_r:

$$t_r = t - \frac{|r - \omega(t_r)|}{c}$$

$$\boldsymbol{\mathscr{r}} = r - \omega(t_r)$$

Only one point contributes to the potentials at any particular time *t*. Otherwise would violate the special relativity. (Details can be found in Griffiths, 4th ed., page 453 and 454.)

$$V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{qc}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\vartheta} \qquad \text{Eq. 10.46}$$

where ϑ is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point **r**.

Moreover, since the current density is $\rho\vartheta$, the vector potential is:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{qc\vartheta}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\vartheta} = \frac{\vartheta}{c^2}V(\mathbf{r},t) \qquad \text{Eq. 10.47}$$

Eq. 10.46 and Eq. 10.47 are the famous **Lienard-Wiechert potentials** for a moving point charge.

Example problem 10.3: Find the potentials of a point charge moving with constant velocity.

10.3.2 The Fields of a Moving Charge

Using the Lienard-Wiechert potentials we can calculate the fields of a moving point charge, **E** and **B**.

<u>Calculation of E and B</u>

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla V = \frac{qc}{4\pi\varepsilon_0} \frac{-1}{(\boldsymbol{\mathscr{r}}c - \boldsymbol{\mathscr{r}}\vartheta)^2} \nabla(\boldsymbol{\mathscr{r}}c - \boldsymbol{\mathscr{r}}\vartheta)$$

Since $\boldsymbol{\mathscr{r}}$=c(t- t_r), then $\nabla\boldsymbol{\mathscr{r}}$=-c$\nabla t_r$

$$\nabla(\boldsymbol{\mathscr{r}}\vartheta) = (\boldsymbol{\mathscr{r}}\nabla)\vartheta + \vartheta(\nabla\boldsymbol{\mathscr{r}}) + \boldsymbol{\mathscr{r}} \times (\nabla \times \vartheta) + \vartheta \times (\nabla \times \boldsymbol{\mathscr{r}})$$

$$(\vartheta \cdot \nabla)\vec{\vartheta} = \left(\vartheta_x \frac{\partial}{\partial x} + \vartheta_y \frac{\partial}{\partial y} + \vartheta_z \frac{\partial}{\partial z}\right) \vartheta(t_r)$$

$$= \vartheta_x \frac{d\vartheta}{dt_r}\frac{\partial t_r}{\partial x} + \vartheta_y \frac{d\vartheta}{dt_r}\frac{\partial t_r}{\partial y} + \vartheta_z \frac{d\vartheta}{dt_r}\frac{\partial t_r}{\partial z}$$

$$= \vec{a}(\vec{\vartheta} \nabla t_r)$$

$$\boldsymbol{\vartheta}\nabla(\boldsymbol{\mathscr{r}}) = (\boldsymbol{\vartheta}\nabla)\boldsymbol{r} - (\boldsymbol{\vartheta}\nabla)\boldsymbol{\omega}$$

$$(\boldsymbol{\vartheta}\nabla)\boldsymbol{r} = \left(\vartheta_x \frac{\partial}{\partial x} + \vartheta_y \frac{\partial}{\partial y} + \vartheta_z \frac{\partial}{\partial z}\right) (x\hat{x} + y\hat{y} + z\hat{z})$$

$$= \vartheta_x \hat{x} + \vartheta_y \hat{y} + \vartheta_z \hat{z} = \boldsymbol{\vartheta}$$

$$(\vartheta.\nabla)\omega = \vartheta(\vartheta.\nabla\, t_r)$$

$$\vec{\nabla}\times\vec{\vartheta} = \overrightarrow{-a}\times\vec{\nabla}t_r$$

$$\vec{\nabla}\times\vec{\boldsymbol{\mathscr{r}}} = \vec{\vartheta}\times\vec{\nabla}t_r$$

$$\begin{aligned}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) &= \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r) + \mathbf{v} - \mathbf{v}(\mathbf{v}\cdot\nabla t_r) - \boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r) + \mathbf{v}\times(\mathbf{v}\times\nabla t_r)\\ &= \mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a} - v^2)\nabla t_r\end{aligned}$$

$$\nabla V = \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\left[\mathbf{v} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\nabla t_r\right]$$

To complete the calculation, we need to know ∇t_r.

$$-c\nabla t_r = \nabla\mathscr{r} = \nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}} = \frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})$$

$$= \frac{1}{\mathscr{r}}\left[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})\right]$$

$$\nabla t_r = \frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\vartheta}$$

$$\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\vartheta)^2} [\vartheta + (c^2 - \vartheta^2 + \boldsymbol{\mathscr{r}}a)\nabla t_r]$$

$$\nabla \mathrm{V} = \frac{qc}{4\pi\varepsilon_0} \frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\vartheta)^3} [(\mathscr{r}c - \boldsymbol{\mathscr{r}}\vartheta)\vartheta - (c^2 - \vartheta^2 + \boldsymbol{\mathscr{r}}a)\boldsymbol{\mathscr{r}}]$$

With a similar calculation,

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3} \left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\mathbf{v} \right]$$

Introducing the vector u, $\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \left[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})\right]$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t).$$

- If $\boldsymbol{\vartheta}$ and $\boldsymbol{a}$ are both zero, $\boldsymbol{E}(r,t)$ reduces to the old electrostatic result.
- The first term in E falls off as the inverse square of distance from the particle. For this reason, it is sometimes called the **generalized Coulomb field**.
- The second term falls of as the inverse first power of $\boldsymbol{\mathscr{r}}$ and is therefore dominant at large distances. It is this term that is responsible for electromagnetic radiation. Accordingly, it is called the **radiation field.**
- The magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.

The electric and magnetic fields of a point charge moving with constant velocity are shown in the Figure 10.10 and 10.11, respectively.

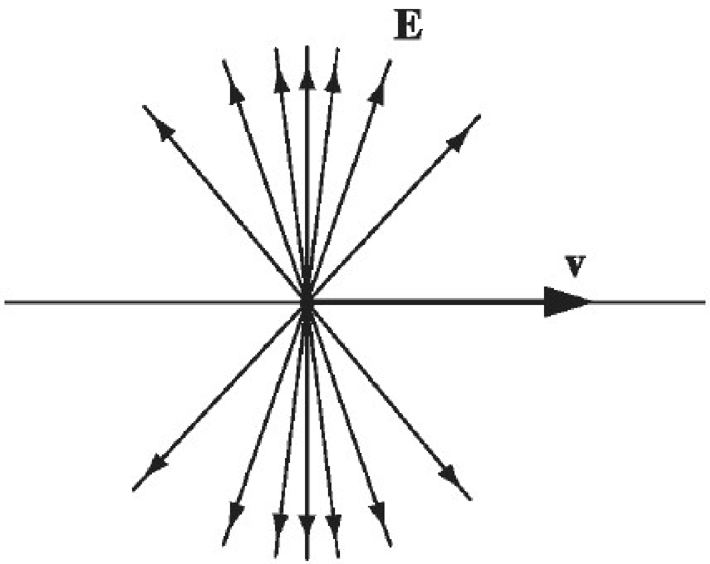

Figure 10.10

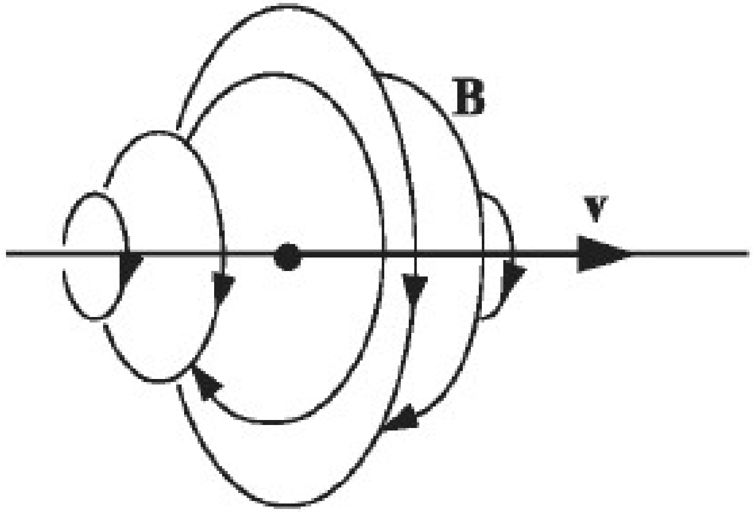

Figure 10.11

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.$$

$$\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

- The field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion because of the $sin^2\theta$ in the denominator.

- In the forward and backward directions ***E*** is *reduced* by a factor $(1 - \vartheta^2/c^2)$ relative to the field of a charge at rest.

- In the perpendicular direction ***E*** is *enhanced* by a factor $1/\sqrt{1 - \vartheta^2/c^2}$.

- Lines of ***B*** *circle around* the charge.

- When $\vartheta^2/c^2 \ll 1$, the equations reduce to the Coulomb's law and the Biot-Savart law for an electric and magnetic field of a point charge, respectively.