# 4.1 Polar Coordinates

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the polar coordinate system, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled $O$. Then we draw a ray (half-line) starting at $O$ called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

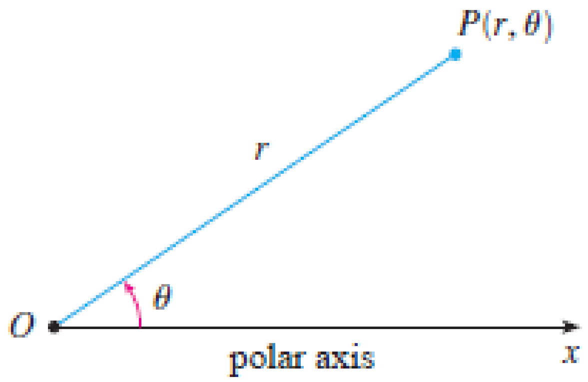



**FIGURE 1**

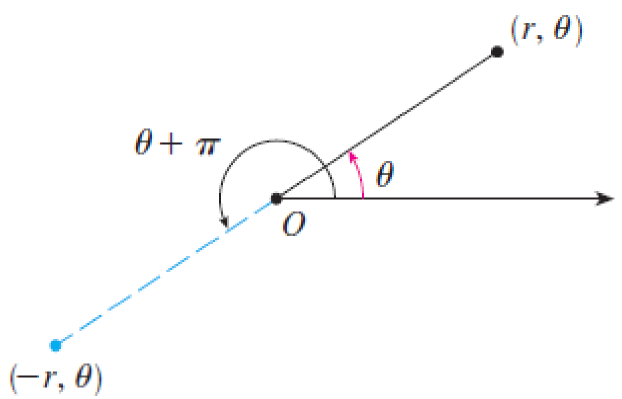



**FIGURE 2**

If $P$ is any other point in the plane, let $r$ be the distance from $O$ to $P$ and let $\theta$ be the angle between the polar axis and the line $OP$ as in Figure 1. Then the point $P$ is represented by the ordered pair $(r, \theta)$ and $r, \theta$ are called **polar coordinates** of $P$. We use the convention that an angle r is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of $\theta$.

We extend the meaning of polar coordinates $(r, \theta)$ to the case in which $r$ is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and $(r, \theta)$ lie on the same line through $O$ and at the same distance $|r|$ from $O$ , but on opposite sides of $O$.

If , $r > 0$ the point $(r, \theta)$ lies in the same quadrant as $\theta$; if $r < 0$, it lies in the quadrant on the opposite side of the pole.

Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$ .

Example 1 Plot the points whose polar coordinates are given.

$$(\boldsymbol{a})\ \left(1, \frac{5\pi}{4}\right), (\boldsymbol{b})\ (2, 3\pi), (\boldsymbol{c})\ \left(2, -\frac{2\pi}{3}\right), (\boldsymbol{d}) \left(-3, \frac{3\pi}{4}\right)$$

## Solution

The points are plotted in Figure 3.

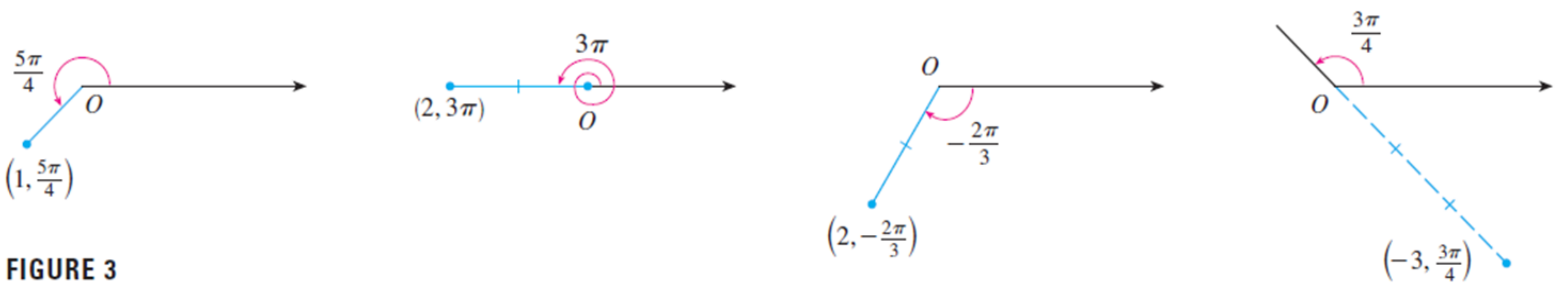



FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $\left(1, \frac{5\pi}{4}\right)$ in Example 1(a) can be written as $\left(1, \frac{-3\pi}{4}\right)$ or $\left(1, \frac{13\pi}{4}\right)$ or $\left(-1, \frac{\pi}{4}\right)$

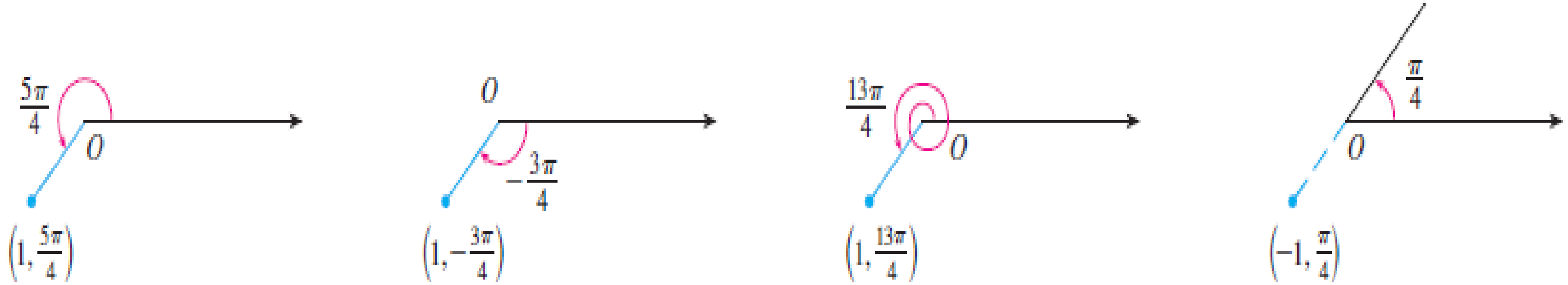



FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle $2\pi$ , the point represented by polar coordinates $(r, \theta\ )$ is also represented by

$$(r, \theta + 2n\pi\ ) \text{ and } (-r, \theta + (2n + 1)\pi\ )$$

where $n$ is any integer.

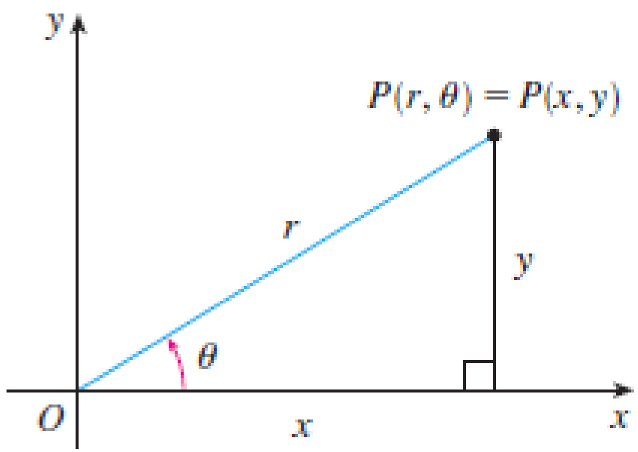



FIGURE 5

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point has Cartesian coordinates $(x, y)$ and polar coordinates $(r, \theta)$ , then, from the figure, we have

$$x = r\cos\theta\ ,\ y = r\sin\theta\ \ldots(1)$$

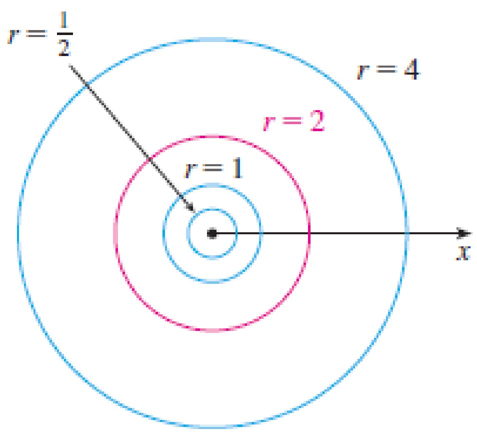
Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find $r$ and $\theta$ when $x$ and $y$ are known, we use the equations

$$r^2 = x^2 + y^2 \,, \tan\theta = \frac{y}{x} \ldots (2)$$

# Polar Curves

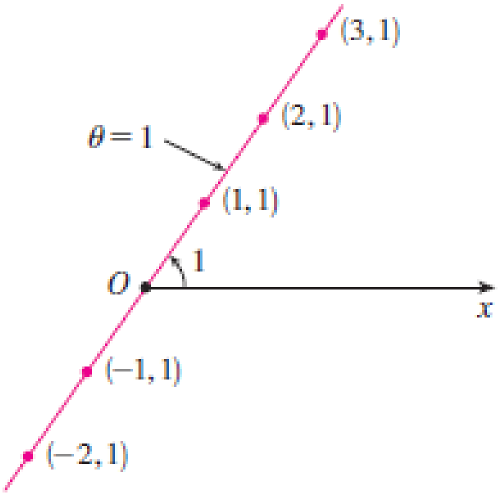
The graph of a polar equation $r = f(\theta)$, or more generally , $F(r, \theta)$ consists of all points $P$ that have at least one polar representation $(r,\theta)$ whose coordinates satisfy the equation.

Example: What curve is represented by the polar equation $r = 2$?




The curve consists of all points $(r,\theta)$ with $r = 2$. Since $r$ represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center $O$ and radius 2. In general, the equation $r = a$ represents a circle with center $O$ and radius $|a|$.

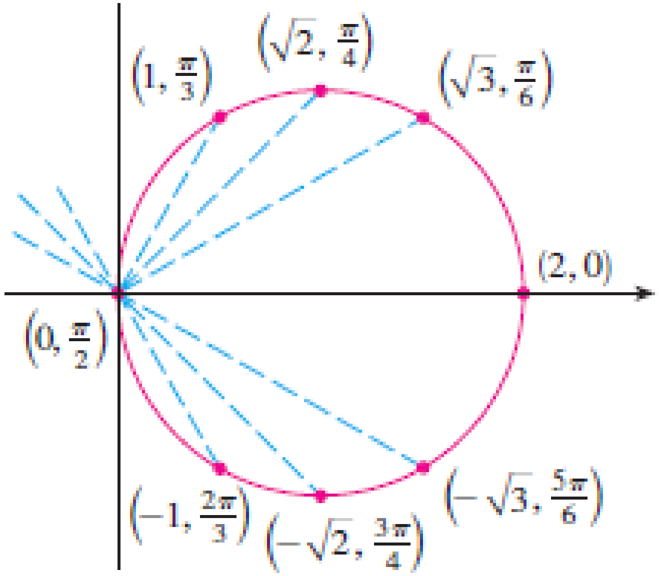
Example: Sketch the polar curve $\theta = 1$.




This curve consists of all points $(r,\theta)$ such that the polar angle $\theta$ is 1 radian. It is the straight line that passes through $O$ and makes an angle of 1 radian with the polar axis as in the Figure. Notice that the points $(r,1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

Example: Sketch the polar curve $r = 2cos\theta$.

| $\theta$ | $r = 2\cos\theta$ |
|---|---|
| 0 | 2 |
| $\pi/6$ | $\sqrt{3}$ |
| $\pi/4$ | $\sqrt{2}$ |
| $\pi/3$ | 1 |
| $\pi/2$ | 0 |
| $2\pi/3$ | $-1$ |
| $3\pi/4$ | $-\sqrt{2}$ |
| $5\pi/6$ | $-\sqrt{3}$ |
| $\pi$ | $-2$ |




We find the values of $r$ for some convenient values of $\theta$ and plot the corresponding points $(r,\theta)$. Then we join these points to sketch the curve, which appears to be a circle. We have used only values of $\theta$ between 0 and $\pi$ , since if we let $\theta$ increase beyond $\pi$, we obtain the same points again.

# Areas and Lengths in Polar Coordinates

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

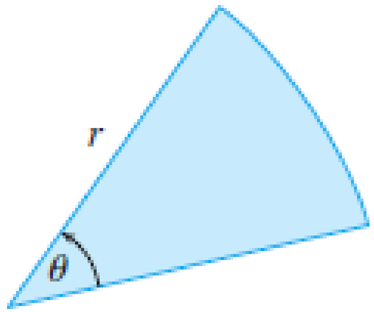



FIGURE 1

$$A = \frac{1}{2} r^2 \theta \ldots (1)$$

where, as in Fig 1 $r$ is the radius and $\theta$ is the radian measure of the central angle.

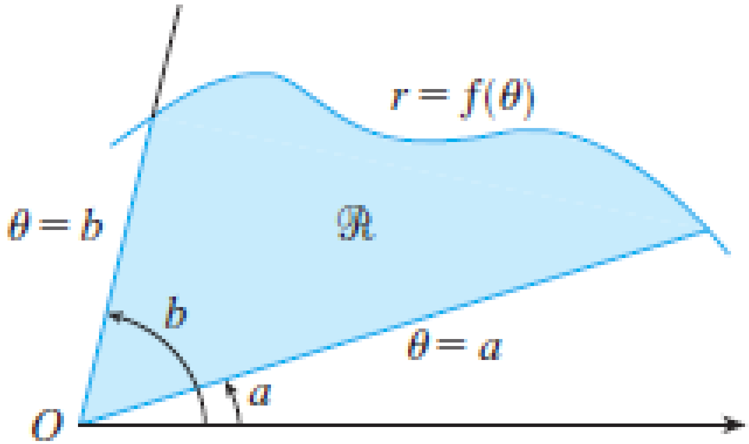



FIGURE 2

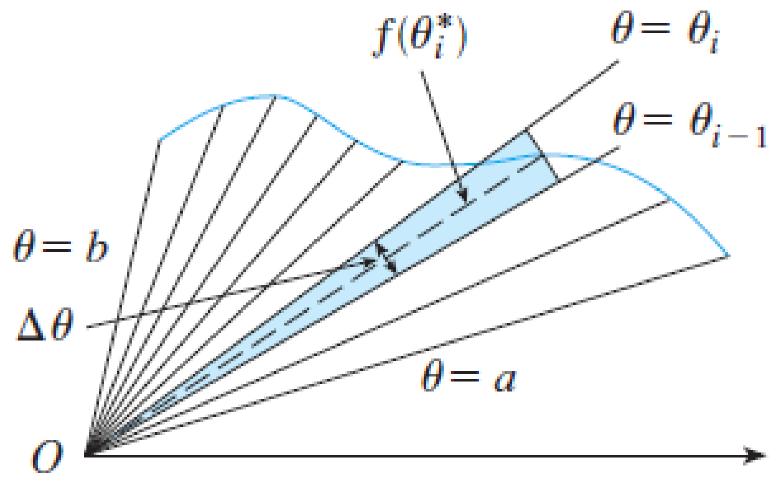



FIGURE 3

Example-1 Find the area enclosed by one loop of the four-leaved rose $r = cos2\theta$.

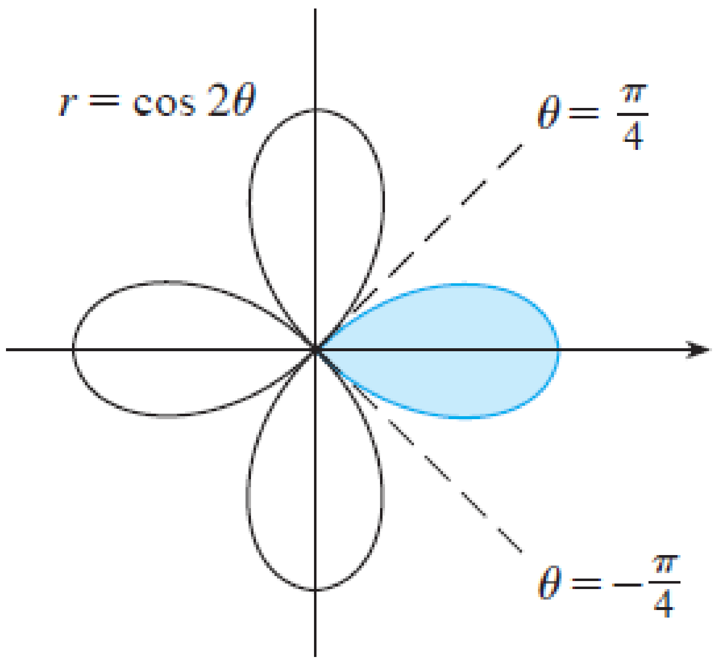



**FIGURE 4**