TERS LAPLACE DÖNÜŞÜMLERİ

$t \geq 0$ için $f(t)$ fonksiyonunun Laplace dönüşümü

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

ile tanımlıdır. $F(s)$ fonksiyonunun ters Laplace dönüşümü

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

dir.

Teorem 1 (Lineerlik Özelliği): $\mathcal{L}^{-1}\{F_i(s)\}$, $i = 1, 2, ..., n$, mevcut olsun. $c_1,\ c_2, ..., c_n$ 'ler keyfi sabitler olmak üzere

$$\mathcal{L}^{-1}\{c_1F_1(s) + c_2F_2(s) + ... + c_nF_n(s)\} = c_1\mathcal{L}^{-1}\{F_1(s)\}+c_2\mathcal{L}^{-1}\{F_2(s)\}+...+c_n\mathcal{L}^{-1}\{F_n(s)\}$$

dir.

Teorem 2: $\mathcal{L}^{-1}\{F(s)\} = f(t)$ olsun.

i) $\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$ dir.

ii) $\mathcal{L}^{-1}\left\{F^{(n)}(s)\right\} = (-1)^n t^n f(t)$ dir.

iii) $\mathcal{L}^{-1}\{e^{-as}F(s)\} = u_a(t) f(t-a)$ dır.

Örnek 1. $\mathcal{L}^{-1}\left\{\dfrac{s-2}{s^2+4s+12}\right\} =?$

Çözüm.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+4s+12}\right\} &= \mathcal{L}^{-1}\left\{\frac{s+2-4}{(s+2)^2+\left(2\sqrt{2}\right)^2}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+\left(2\sqrt{2}\right)^2}\right\} - \sqrt{2}\mathcal{L}^{-1}\left\{\frac{2\sqrt{2}}{(s+2)^2+\left(2\sqrt{2}\right)^2}\right\} \\
&= e^{-2t}\cos\left(2\sqrt{2}t\right) - \sqrt{2}e^{-2t}\sin\left(2\sqrt{2}t\right)
\end{aligned}$$

Örnek 2. $\mathcal{L}^{-1}\left\{\dfrac{2s-3}{s(s-2)(s^2-2s+5)}\right\} =?$

Çözüm. Basit kesirlerine ayırma yönteminden

$$\frac{2s-3}{s(s-2)(s^2-2s+5)} = \frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2-2s+5}$$

olarak yazılırsa $A=\frac{3}{10},\ B=\frac{1}{10},\ C=-\frac{2}{5},\ D=\frac{3}{5}$ olur. Buradan

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{2s-3}{s\left(s-2\right)\left(s^2-2s+5\right)}\right\} \\
=\ &\frac{3}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{-2s+3}{\left(s-1\right)^2+2^2}\right\} \\
=\ &\frac{3}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{2}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{\left(s-1\right)^2+2^2}\right\}+\frac{1}{10}\mathcal{L}^{-1}\left\{\frac{2}{\left(s-1\right)^2+2^2}\right\} \\
=\ &\frac{3}{10}+\frac{1}{10}e^{2t}-\frac{2}{5}e^{t}\cos 2t+\frac{1}{10}e^{t}\sin 2t
\end{aligned}$$

elde edilir.

Konvolüsyon Operatörü

$f,g:\mathbb{R}\to\mathbb{R}$ fonksiyonlarının konvolüsyonu

$$\left(f*g\right)\left(x\right)=\int\limits_0^x f\left(t\right)g\left(x-t\right)dt$$

olarak tanımlanır. Konvolüsyon operatörü $\left(f*g\right)\left(x\right)=\left(g*f\right)\left(x\right)$ değişme özelliğine sahiptir.

Teorem 3 (Konvolüsyon Teoremi): $\mathcal{L}\left\{f\left(t\right)\right\}=F\left(s\right)$ ve $\mathcal{L}\left\{g\left(t\right)\right\}=G\left(s\right)$ ise

$$\mathcal{L}\left\{\left(f*g\right)\left(t\right)\right\}=\mathcal{L}\left\{f\left(t\right)\right\}\mathcal{L}\left\{g\left(t\right)\right\}=F\left(s\right)G\left(s\right)$$

dir. Ayrıca

$$\mathcal{L}^{-1}\left\{F\left(s\right)G\left(s\right)\right\}=\left(f*g\right)\left(t\right)$$

dir.

Örnek 3. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-s}\right\}=?$

Çözüm. $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1$ ve $\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}=e^x$ olduğundan konvolüsyon teoreminden

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2-s}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s-1}\right\}=e^x*1 \\
&= \int\limits_0^x e^t dt=e^x-1
\end{aligned}$$

elde edilir.

Örnek 4. $\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+1\right)}\right\}=?$

Çözüm. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=x$ ve $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin x$ olduğundan konvolüsyon teoreminden

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+1\right)}\right\} &= \sin x * x \\
&= \int\limits_0^x (x-t)\sin t dt \\
&= \left.\left(-(x-t)\cos t-\sin t\right)\right|_0^x \\
&= -\sin x + x
\end{aligned}$$

elde edilir.