EEE328
Digital Signal Processing

Ankara University

Faculty of Engineering

Electrical and Electronics Engineering Department

Transform Analysis of LTI Systems

EEE328 Digital Signal Processing

Lecture 12

Agenda

- Transform Analysis of LTI Systems
- The Frequency Response of LTI Systems
- System Functions
- Relationship Between Magnitude and Phase
- All-Pass Systems
- Minimum Phase and All-Pass Decomposition

Transform Analysis of LTI Systems

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\mathcal{Z} \Big\updownarrow$$

$$Y(z) = H(z)X(z)$$

The Frequency Response of LTI Systems

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

$$\measuredangle Y(e^{j\omega}) = \measuredangle H(e^{j\omega}) + \measuredangle X(e^{j\omega})$$

Phase Distortion and Delay

$$h_{id}[n] = \delta[n - n_d]$$

Ideal Delay System

$$H_{id}[n] = e^{-j\omega n_d}$$

$$\left|H_{id}\left(e^{j\omega}\right)\right| = 1$$

$$\sphericalangle H\left(e^{j\omega}\right) = -\omega n_d, \qquad |\omega| < \pi$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\mathcal{Z} \Updownarrow$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} Y(z)$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

Stability

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

$$\sum_{k=-\infty}^{\infty} |h[n]z^{-n}| < \infty$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

Inverse Systems

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

Inverse Systems

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

If $H(z)$ is causal, its ROC: $|z| > \max_k |d_k|$

$$H_i(z) = \left(\frac{a_0}{b_0}\right)\frac{\prod_{k=1}^{M}(1 - d_k z^{-1})}{\prod_{k=1}^{N}(1 - c_k z^{-1})}$$

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

A First-Order Ifinite Impulse Response (IIR) System

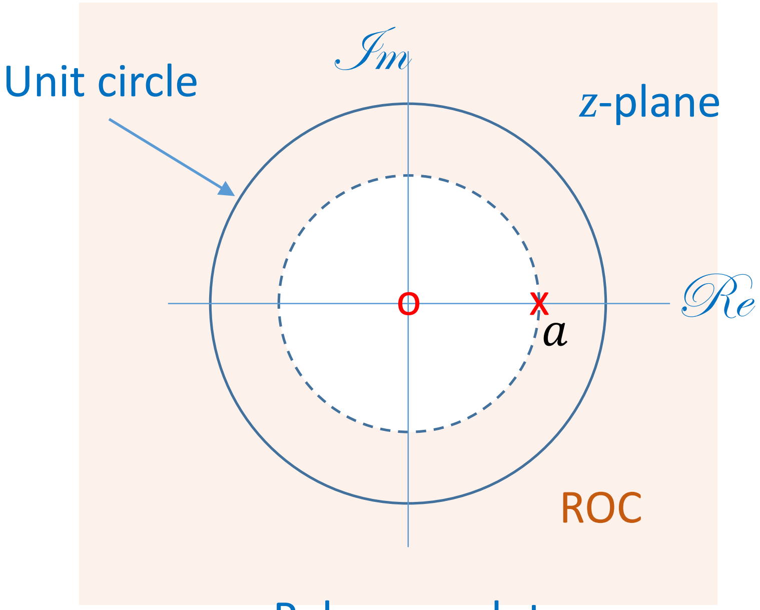
$$y[n] - ay[n-1] = x[n]$$

$$H(z) = \frac{1}{1 - az^{-1}}$$ System Function

ROC: $|z| > |a|$

Condition for Stability: $|a| < 1$

$$h[n] = a^n u[n]$$ Impulse Response



Pole-zero plot

System Functions for Systems Characterized by Linear Constant-Coefficient Difference Equations

M=7

A Simple Finite Impulse Response (FIR) System

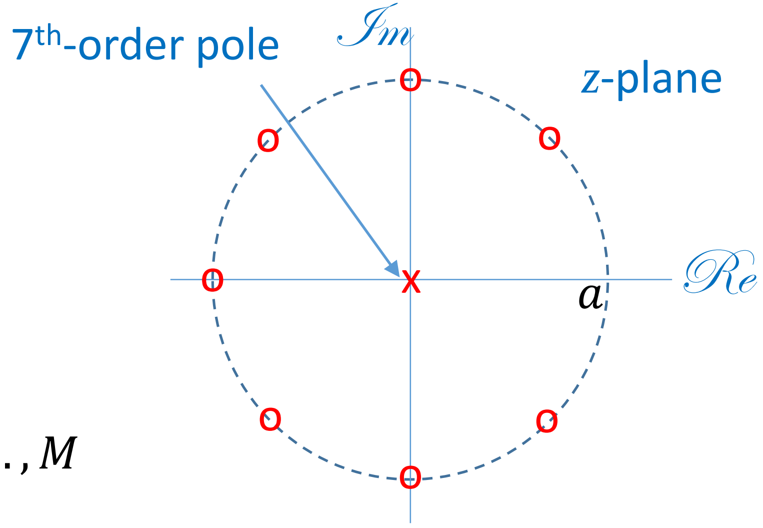
$$h[n] = \begin{cases} a^n, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - a z^{-1}}$$

Zeros: $z_k = a e^{-j2\pi k/(M+1)}, \; k = 0,1,\ldots,M$

a: Real and positive

The pole at $z = a$ is cancelled by a zero.



Pole-zero plot

ROC: $|z| > |a|$

Relationship Between Magnitude and Phase

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*)|_{z=e^{j\omega}}$$

$$H(z) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

$$H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k^* z)}$$

$$C(z) = H(z)H^*(1/z^*) = \left(\frac{b_0}{a_0}\right)\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k^* z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k^* z)}$$

All-Pass Systems

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$H_{ap}(e^{j\omega}) = \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} = e^{-j\omega}\frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}}$$

$$H_{ap}(z) = A\prod_{k=1}^{M_r}\frac{z^{-1} - d_k}{1 - d_k z^{-1}}\prod_{k=1}^{M_c}\frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

Minimum-Phase and All-Pass Decomposition

$$H(z) = H_{min}(z)H_{ap}(z)$$

$$H(z) = H_1(z)(z^{-1} - c^*)$$

$$H(z) = H_1(z)(1 - cz^{-1})\frac{z^{-1} - c^*}{1 - cz^{-1}}$$

References

- Signals & Systems, Second Edition, A. V. Oppenheim, A. S. Willsky with S. H. Nawab, Prentice Hall, 1997
- Discrete-Time Signal Processing, Second Edition, A. V. Oppenheim, R. W. Schafer with J. R. Buck, Prentice Hall, 1999