Lecture 5 : **Relativistic kinematics -1**

Lorentz transformations : Let us imagine two reference frames S and S' with spacetime coordinates (ct, x, y, z) and (ct', x', y', z') respectively. For simplicity first consider the simple case in which the frame S' moves along the x-direction with a velocity v along x-axis with respect to the frame S such that their coordinate axes remain parallel to each other. Lorentz transformations connect the spacetimes of these two frames via

i. $x' = \gamma(x - vt)$

ii. $y' = y$

iii. $z' = z$

iv. $t' = \gamma\left(t - \frac{v}{c^2}x\right)$

where the gamma factor is given as :

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Lorentz transformations have several important immediate consequences :

a) Simultaneity is frame dependent : If two events occur simultaneously in frame S at differenet locations then these two events are not simultaneous in a frame S' moving w.r.t. the first one.

b) Lorentz length contraction : The length of a ruler in a rest frame is different than the length of the same ruler measured by a moving observer. Moving observers always measure a (contracted) value which is smaller by a factor of gamma.

c) Time dilation : Time flows at different rate w.r.t.different frames moving at different speeds. As an example the average lifetime of the muon in its rest frame is 2.2 microseconds. The lifetime measured by a moving observer is larger the this value by a factor of gamma. This rusult explains why we are able to detect the atmospheric muons on the Earth surface.

d) Velocity addition rule differs from that of the Galilean relation :

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)}$$

Four vectors : Position four vector is defined as follows :

$x^\mu, \mu = 0, 1, 2, 3$ with the contravariant components

$$x^0 = ct, \qquad x^1 = x, \qquad x^2 = y, \qquad x^3 = z$$

It transforms under Lorentz transformation of the special case above via :

$$x^{0\prime} = \gamma(x^0 - \beta x^1)$$

$$x^{1\prime} = \gamma(x^1 - \beta x^0)$$

$$x^{2\prime} = x^2$$

$$x^{3\prime} = x^3$$

where

$$\beta \equiv \frac{v}{c}$$

A more compact tensor notation is :

$$x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \qquad (\mu = 0, 1, 2, 3)$$

One can easily show that the following quantity remains invariant under Lorentz transformations in going from S to S' :

$$I \equiv (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0'})^2 - (x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2$$

By introducing a flat spacetime metric g,

$$\mathbf{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

the above invariant can be expressed as

$$I = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu = g_{\mu\nu} x^\mu x^\nu$$

Covariant components of position vector are given by

$$x_\mu \equiv g_{\mu\nu} x^\nu$$

so that the above invariant can also be written as

$$I = x_\mu x^\mu$$

note that summation over repeated index is always implied.
Four velocity is defined to be the derivative of the position four vector with respect to proper time.

Scalar product is defined to be

$$a \cdot b \equiv a_\mu b^\mu$$

so that it becomes

$$a \cdot b = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$$

Obviously

$$a^2 \equiv a \cdot a = (a^0)^2 - \mathbf{a}^2$$

needs not to be positive. Hence we classify the spacetime distances into three groups :

$$\begin{array}{ll} \text{If } a^2 > 0, & a^\mu \text{ is called } \textit{timelike} \\ \text{If } a^2 < 0, & a^\mu \text{ is called } \textit{spacelike} \\ \text{If } a^2 = 0, & a^\mu \text{ is called } \textit{lightlike} \end{array}$$

Because the second postulate of the special relativity brings an upper limit for the speed of the physical objects, c, which is the speed of the light the spacelike intervals are not physically reachable. Two spacetime points separated in spacelike manner can not communicate with eachother.

Four velocity : In order to preserve the Lorentz tarnsformation properties of the four vectors it is defined as the derivative of the position four vector with respect tho the proper time (namely rest frame time which is an invariant)

$$\eta^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

so that

$$\eta_\mu \eta^\mu = c^2$$

Energy momentum four vector is simply given by :

$$p^\mu = m\eta^\mu$$

Homework

Solve the following problems at the end of the Chapter III of the textbook by D.Griffiths “Introduction to Elementary Particles J.Wiley)

Solve Problem 3.1

Solve Problem 3.2

Solve Problem 3.3

Solve Problem 3.4

Solve Problem 3.5

Solve Problem 3.6