Lecture 12 : Dirac Equation -1

Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

$$\mathbf{p} \to -i\hbar\nabla, \qquad E \to i\hbar\frac{\partial}{\partial t}$$

Replace these operators into the relativistic energy formula

$$E^2 - \mathbf{p}^2c^2 = m^2c^4:$$

One gets the Klein-Gordon equation :

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$

On the other hand in four-vector notation :

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

If we replace it in the relativistic energy momentum relation

$$p^\mu p_\mu - m^2c^2 = 0$$

One may easily get the Klein-Gordon wave equation :

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = \left(\frac{mc}{\hbar}\right)^2\psi$$

which is second order also in time.

In order to obtain a first order relativistic wave eqaution Dirac tried to factor the energy-momentum relation

$$p^\mu p_\mu - m^2c^2 = 0$$

$$(p^{\mu}p_{\mu} - m^2c^2) = (\beta^{\kappa}p_{\kappa} + mc)(\gamma^{\lambda}p_{\lambda} - mc)$$

One gets the Dirac equation

$$\gamma^{\mu}p_{\mu} - mc = 0$$

Dirac matrices γ^{μ} satisfy the Dirac algebra :

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$$

where $g^{\mu\nu}$ is the Minkowski metric and the gamma matrices have the representations :

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where σ^i are the 2x2 Pauli matrices.

The wavefunction is the Dirac spinor with four complex elements in general.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

For momentum zero case, $\vec{p}=0$ one gets four independent solutions :

$$\psi^{(1)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi^{(2)} = e^{-i(mc^2/\hbar)t}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi^{(3)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi^{(4)} = e^{+i(mc^2/\hbar)t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

For non-zero three momentum $\vec{p} \neq 0$ case :

First express the Dirca spinor as composed of two two-spinors :

$$\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \qquad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

The four independent solutions are :

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : u_B = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{p^0 + mc} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E + mc^2} \begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : u_B = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{p^0 + mc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E + mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

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$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} : u_A = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{p^0 + mc} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E + mc^2} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

The first two of them are the particle solutions with spin up and down respectively. The other two represent the anti-particle solutions.

The bilinear covariants are :

$$\overline{\psi}\psi \qquad \overline{\psi}\gamma^5\psi \qquad \overline{\psi}\gamma^\mu\psi \qquad \overline{\psi}\gamma^\mu\gamma^5\psi \qquad \overline{\psi}\sigma^{\mu\nu}\psi$$

which have scalar, pseudoscalar, vector, pseudovector and tensor characters under transformations. Here $\sigma^{\mu\nu}$ matrices are :

$$\sigma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

Homework :

Solve the following problems from the textbook D.Griffiths' "Int. to Elementary Particles"

Solve Problem 7.1

Solve Problem 7.2

Solve Problem 7.3

Solve Problem 7.4

Solve Problem 7.5