Lecture 13 : **Basics of Gauge Theories -1**

Lagrangian mechanics :

Euler-Lagrange equations :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Lagrangians in field teory - Klein Gordon and Dirac equations :

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = \frac{\partial \mathcal{L}}{\partial \phi_i}$$

a) Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2$$

It leads to the following relativistic wave equation for spin zero particle with mass m :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \phi^2$$

b) Dirac Lagrangian

$$\mathcal{L} = i(\hbar c)\bar{\psi}\gamma^\mu \partial_\mu \psi - (mc^2)\bar{\psi}\psi$$

The corresponding Euler-Lagrange equation is the Dirac equation for massive spin ½ particle :

$$i\gamma^\mu \partial_\mu \psi - \left(\frac{mc}{\hbar}\right)\psi = 0$$

Also the adjoint spinor satisfies the following equation :

$$i\,\partial_\mu \bar{\psi}\gamma^\mu + \left(\frac{mc}{\hbar}\right)\bar{\psi} = 0$$

Proca Lagrangian for spin-1 field :

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi}\left(\frac{mc}{\hbar}\right)^2 A^\nu A_\nu$$

And it gives as Euler-Lagrange equation

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \left(\frac{mc}{\hbar}\right)^2 A^\nu = 0$$

Also the Maxwell Lagrangian is :

$$\mathcal{L} = \frac{-1}{16\pi} F^{\mu\nu}F_{\mu\nu} - \frac{1}{c} J^\mu A_\mu$$

It leads to the ingomegeneous Maxwell equations in covarien form :

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

Local gauge invariance :

$$\mathcal{L} = i\hbar c\bar{\psi}\gamma^\mu \partial_\mu\psi - mc^2\bar{\psi}\psi$$

The above Dirac lagrangian Remains invariant under global transformations of the phase of the -Dirac spinor :

$$\psi \longrightarrow e^{i\theta}\psi$$

where θ is a real number. However the same Lagrangians is not invariant if θ is a function of the spacetime, namely local gauge transformations :

$$\psi \longrightarrow e^{i\theta(x)}\psi$$

If we add an interactin term to the Lagrangian then the total Lagrangian becomes invariant under local gauge transformations :

$$\mathcal{L} = [i\hbar c\bar{\psi}\gamma^\mu \partial_\mu\psi - mc^2\bar{\psi}\psi] - (q\bar{\psi}\gamma^\mu\psi)A_\mu$$

Such local transformations on the phase of the field form a U(1) Abelian group.

$$\mathcal{L} = [i\hbar c\bar{\psi}\gamma^\mu \partial_\mu\psi - mc^2\bar{\psi}\psi] + \left[\frac{-1}{16\pi} F^{\mu\nu}F_{\mu\nu}\right] - [(q\bar{\psi}\gamma^\mu\psi)A_\mu]$$

Conclusion : Local gauge symmetry dictates the form of the physical interaction.

Gauge covariant derivative

Observation : If the ordinary derivative in the original Lagrangians is replaced by

$$\mathcal{D}_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu$$

Then local gauge invariance is restored.

For the details examine the textbook D.Griffiths "Int. to Elementary Particles" J.Wiley Chapter 11.

Homework :

Solve the following problems from the textbook D.Griffiths' "Int. to Elementary Particles"

Solve Problem 11.1

Solve Problem 11.2

Solve Problem 11.3

Solve Problem 11.4

Solve Problem 11.5