Electrostatics (source charges are stationary)

What force do source charges exert on test charge Q?

Figure 1

The **principle of superposition**: the interaction between any two charges is completely unaffected by the presence of others.

To determine the force on Q, first compute the force $\mathbf{F}_1$, due to q_1 alone (ignoring all the others); then the force $\mathbf{F}_2$, due to q_2 alone; and so on.
The sum of all individual forces: $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$

the force exerted on a test charge Q due to a single point charge q, is given by **Coulomb's law**:

Figure 2

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}$$

ϵ_0 : the **permittivity of free space.** In SI units, the force is in Newtons (N), distance in meters (m), and charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

$\mathscr{r}$, is the separation vector from $\mathbf{r}'$ (the location of q) to $\mathbf{r}$ (the location of Q):

$$\mathscr{r} = \mathbf{r} - \mathbf{r}'$$

The Electric Field

point charges $q_1, q_2, \ldots, q_n$, at distances $\mathscr{r}_1, \mathscr{r}_2, \ldots, \mathscr{r}_n$ from Q, the total force on Q

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)$$

Figure 3

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\boldsymbol{\mathscr{r}}}_3 + \ldots\right)$$

$$\mathbf{F} = Q\mathbf{E}$$

E is called the **electric field** of the source charges. **E**(**r**) is a function of position (**r**);

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i$$

Example 1. Find the electric field a distance z above the midpoint between two equal charges (q), a distance d apart.

$$E_z = 2\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\cos\theta.$$

Figure 4

$$\mathscr{r} = \sqrt{z^2 + (d/2)^2}$$

$$\cos\theta = z/\mathscr{r}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + (d/2)^2\right]^{3/2}}\,\hat{\mathbf{z}}.$$

When $z >> d$

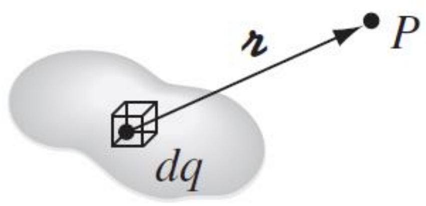
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\,\hat{\mathbf{z}}.$$

(just set $d \rightarrow 0$ in the formula).

Continuous Charge Distributions

If the charge is distributed continuously over some region, the electric field:

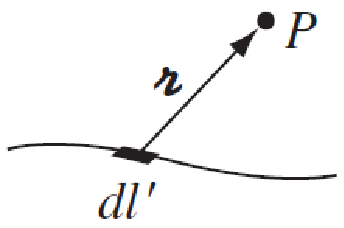
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\mathscr{r}}\, dq$$



Continuous distribution

If the charge is spread out along a *line*, with charge-per-unit-length λ (q/l),

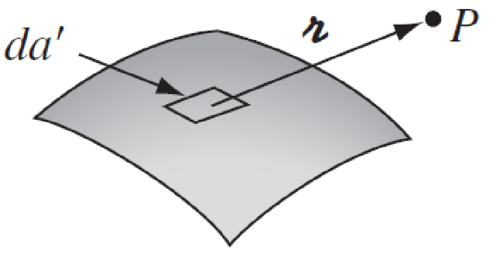
$$dq = \lambda\, dl'$$



Line charge, λ

If the charge is smeared out over a *surface*,
with charge-per-unit-area σ (q/a),

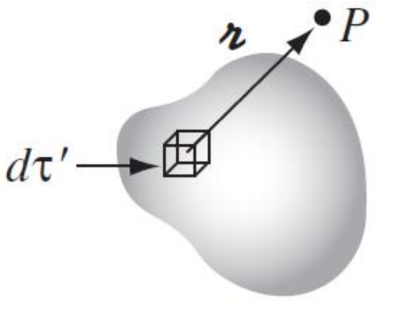
$$dq = \sigma \, da'$$



Surface charge, σ

If the charge fills a *volume*,
with charge-per-unit-volume ρ (q/τ),

$$dq = \rho \, d\tau'$$



Volume charge, ρ

$$dq \to \lambda\, dl' \sim \sigma\, da' \sim \rho\, d\tau'$$

The electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl'$$

The electric field of a surface charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da'$$

The electric field of a volume charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

Example 2. Find the electric field a distance z above the midpoint of a straight line segment of length $2L$ that carries a uniform line charge λ.

$$\left.\begin{aligned} \boldsymbol{\mathscr{r}} &= \mathbf{r} - \mathbf{r}' = z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}} \\ \mathscr{r} &= \sqrt{z^2 + x^2} \end{aligned}\right\} \quad \hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}$$

Figure 6

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dl'$$

$$\begin{aligned} \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\int_{-L}^{L} \frac{\lambda}{z^2 + x^2}\frac{z\,\hat{\mathbf{z}} - x\,\hat{\mathbf{x}}}{\sqrt{z^2 + x^2}}\,dx \\ &= \frac{\lambda}{4\pi\epsilon_0}\left[z\,\hat{\mathbf{z}}\int_{-L}^{L}\frac{1}{(z^2 + x^2)^{3/2}}\,dx - \hat{\mathbf{x}}\int_{-L}^{L}\frac{x}{(z^2 + x^2)^{3/2}}\,dx\right] \\ &= \frac{\lambda}{4\pi\epsilon_0}\left[z\,\hat{\mathbf{z}}\left(\frac{x}{z^2\sqrt{z^2 + x^2}}\right)\Bigg|_{-L}^{L} - \hat{\mathbf{x}}\left(-\frac{1}{\sqrt{z^2 + x^2}}\right)\Bigg|_{-L}^{L}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{2\lambda L}{z\sqrt{z^2 + L^2}}\,\hat{\mathbf{z}}. \end{aligned}$$

$(x = z \tan u, \quad dx = \ldots du\,)$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2 + L^2}} \hat{\mathbf{z}}$$

For points far from the line ($z >> L$),

$$E \cong \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z^2}$$

From far away the line looks like a point charge $q = 2\lambda L$.

In the limit $L \to \infty$, on the other hand, we obtain the field of an infinite straight wire:

$$E = \frac{1}{4\pi\epsilon_0} \frac{2\lambda}{z}$$

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