Problem 7: Find the electric field a distance z from the center of a spherical surface of radius R that carries a uniform charge density σ. Treat the case $z < R$ (inside) as well as $z > R$ (outside). Express your answers in terms of the total charge q on the sphere.

E is in the **z-**direction, uniform charge density σ.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,da' \longrightarrow \sigma da = ?$$

Figure 11

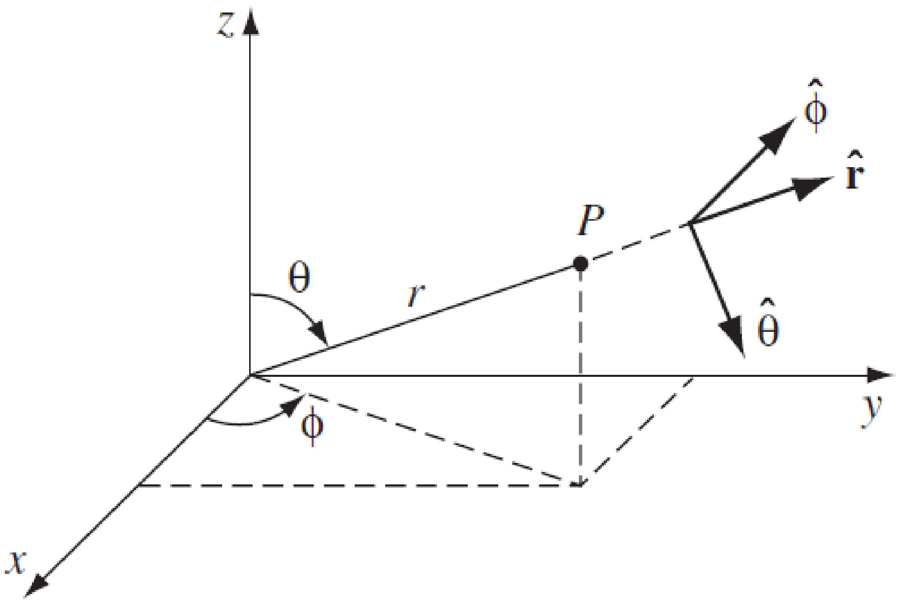
In spherical coordinates:

$$dq = \sigma da = \sigma R^2 \sin\theta \, d\theta \, d\phi$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$$

$$\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$$

Spherical Coordinates



The point P
Cartesian coordinates (x, y, z)
Spherical coordinates (r, θ, φ);

r is the distance from the origin
(the magnitude of the position vector r)

θ (the angle down from the z axis)
is called the **polar angle**,

φ (the angle around from the x axis)
is the **azimuthal angle**.

Any vector **A** can be expressed as

$$\mathbf{A} = A_r\,\hat{\mathbf{r}} + A_\theta\,\hat{\boldsymbol{\theta}} + A_\phi\,\hat{\boldsymbol{\phi}}$$

Transformation of Coordinates

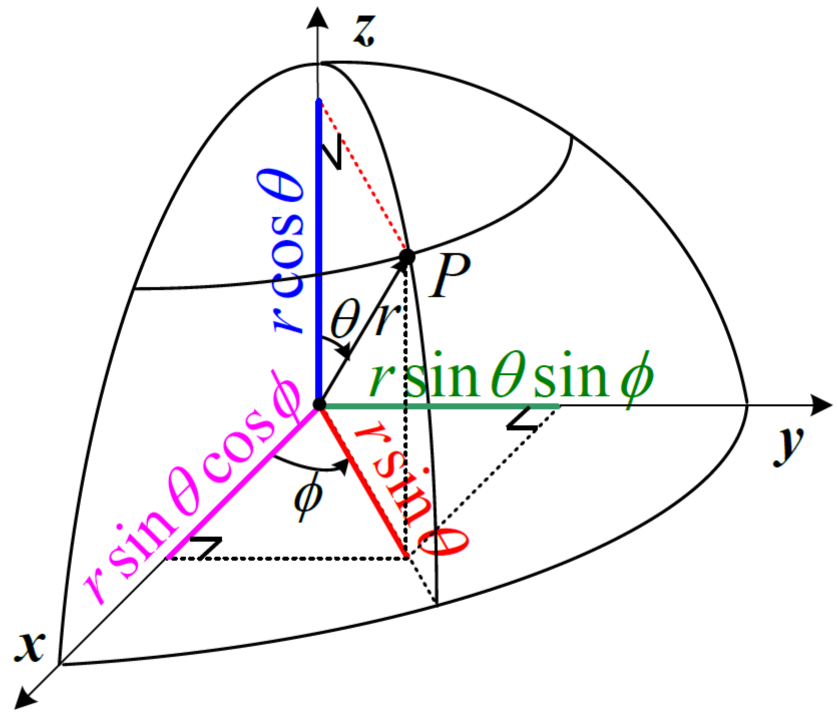
Rectangular ⇔Spherical

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

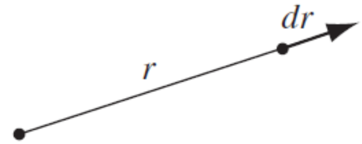


r ranges from 0 to ∞,

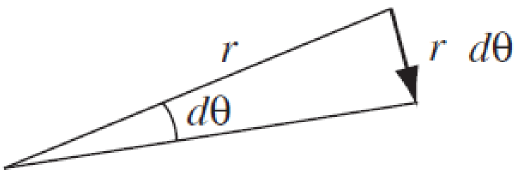
θ from 0 to π

φ from 0 to 2π,

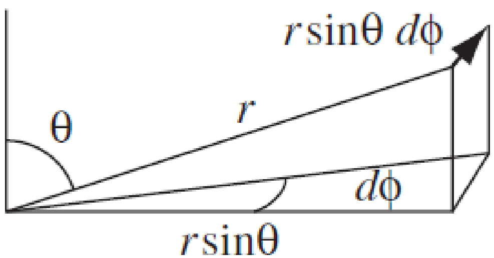
An infinitesimal displacement in the $\hat{\mathbf{r}}$ direction:



$$dl_r = dr$$



$$dl_\theta = r\,d\theta$$



$$dl_\phi = r\sin\theta\,d\phi$$

Thus, the general infinitesimal displacement $d\mathbf{l}$ is

$$d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$$

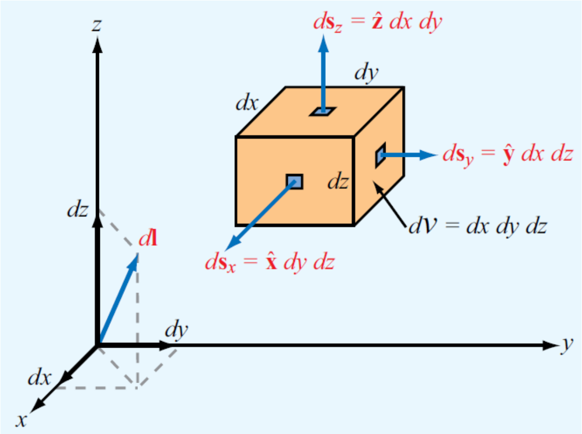
The infinitesimal volume element $d\tau$;

$$d\tau = dl_r\,dl_\theta\,dl_\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

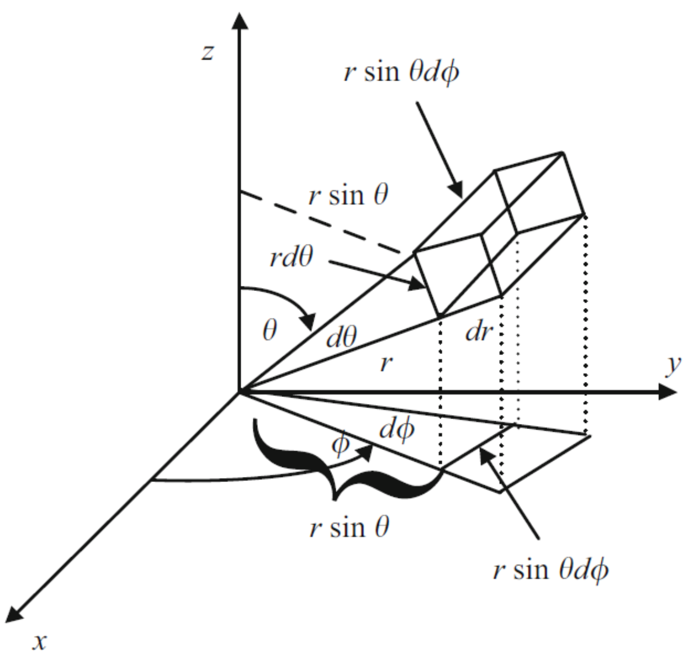
Is there a general expression for *surface* elements $d\mathbf{a}$?

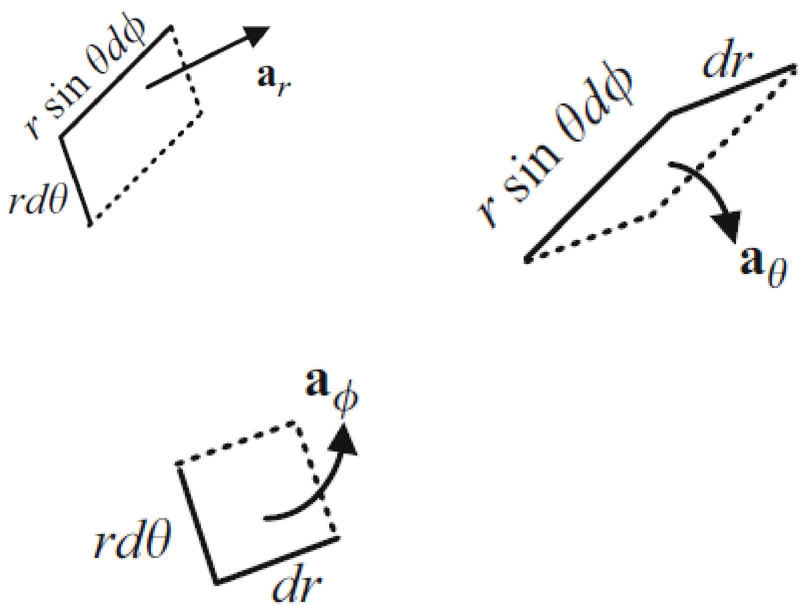
No, it depend on the orientation of the surface.

Differential length, area, and volume
in Cartesian coordinates:



In spherical coordinates:

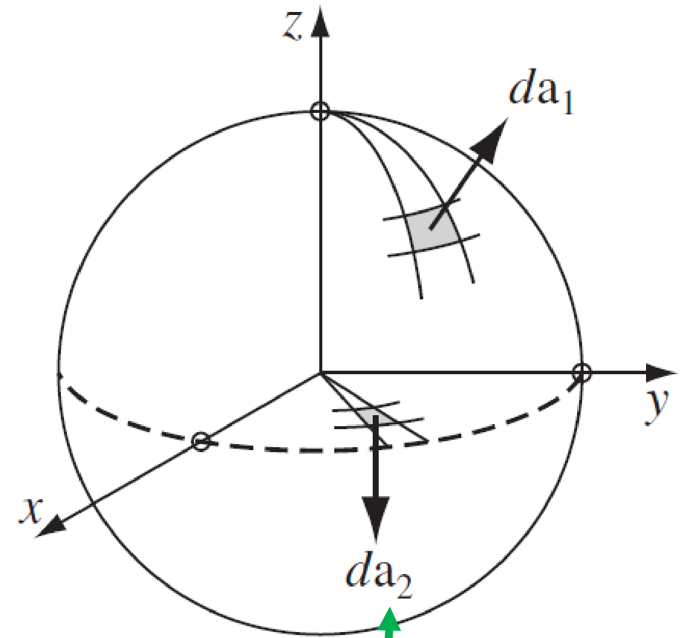




In spherical coordinates:

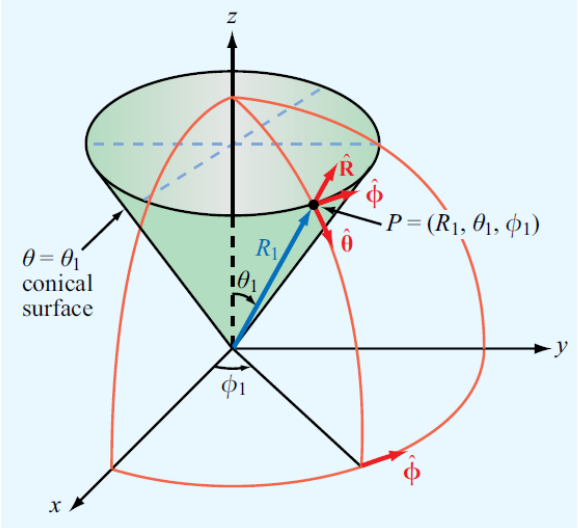
$$dl_r = dr \qquad dl_\theta = r\,d\theta \qquad dl_\phi = r\sin\theta\,d\phi$$

Integrating over the surface of a sphere, then r is constant, whereas θ and φ change (θ–φ spherical surface),



$$d\mathbf{a}_1 = dl_\theta\,dl_\phi\,\hat{\mathbf{r}} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}}$$

For constant θ ; (R–φ conical surface),



$$d\mathbf{a}_2 = dl_r\,dl_\phi\,\hat{\boldsymbol{\theta}} = r\,dr\,d\phi\,\hat{\boldsymbol{\theta}}$$

(For $\theta = \pi/2$; the surface lies in the xy plane)

For constant φ ; (R–θ plane),

The surface area $d\boldsymbol{a} = r\, dr\, d\theta\ \hat{\boldsymbol{\phi}}$

E.g.: Find the volume of a sphere of radius R.

$$V = \int d\tau = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \left(\int_0^R r^2 \, dr \right) \left(\int_0^{\pi} \sin\theta \, d\theta \right) \left(\int_0^{2\pi} d\phi \right)$$

$$= \left(\frac{R^3}{3} \right) (2)(2\pi) = \frac{4}{3} \pi R^3$$

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$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da' \longrightarrow \sigma da = ?$$

Figure 11

By symmetry **E** is in **z**-direction;

$$E(\vec{r}) = E_z\hat{z} \longrightarrow E_z = \left|E(\vec{r})\right|\cos\psi$$

In spherical coordinates:

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$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta$$

$$\cos\psi = \frac{z - R\cos\theta}{\mathscr{r}}$$

$$E_z = \frac{1}{4\pi\epsilon_0} \int\int \frac{\sigma R^2 \sin\theta\, d\theta\, d\phi (z - R\cos\theta)}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \qquad \int d\phi = 2\pi$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2 \sigma) \int_0^{\pi} \frac{(z - R\cos\theta)\sin\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\, d\theta$$

$$\text{Let } u = \cos\theta;\; du = -\sin\theta\, d\theta; \begin{Bmatrix} \theta = 0 \Rightarrow u = +1 \\ \theta = \pi \Rightarrow u = -1 \end{Bmatrix}$$

$$E_z = \frac{1}{4\pi\epsilon_0}(2\pi R^2 \sigma) \int_{-1}^{1} \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}} du$$

$$= \frac{1}{4\pi\epsilon_0}(2\pi R^2 \sigma) \left[\frac{1}{z^2} \frac{zu - R}{\sqrt{R^2 + z^2 - 2Rzu}} \right]_{-1}^{1} = \frac{1}{4\pi\epsilon_0} \frac{2\pi R^2 \sigma}{z^2} \left\{ \frac{(z - R)}{|z - R|} - \frac{(-z - R)}{|z + R|} \right\}$$

For z > R (outside the sphere),

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{4\pi R^2 \sigma}{z^2} = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2}$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{z^2}\, \hat{\mathbf{z}}$$

For z < R (inside),

$$E_z = 0, \qquad \mathbf{E} = \mathbf{0}$$