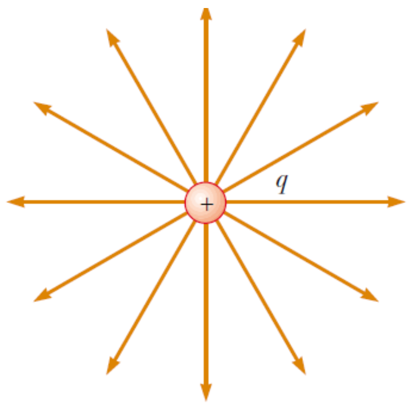
Field Lines, Flux, and Gauss's Law

The electric field of a single point charge q, situated at the origin:

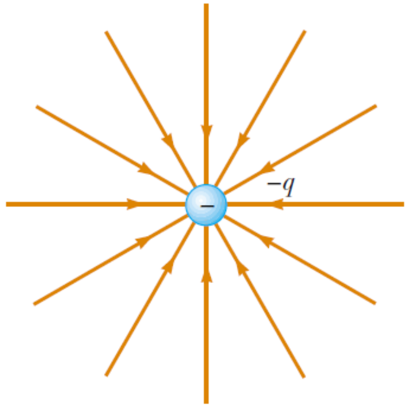
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Figure 12

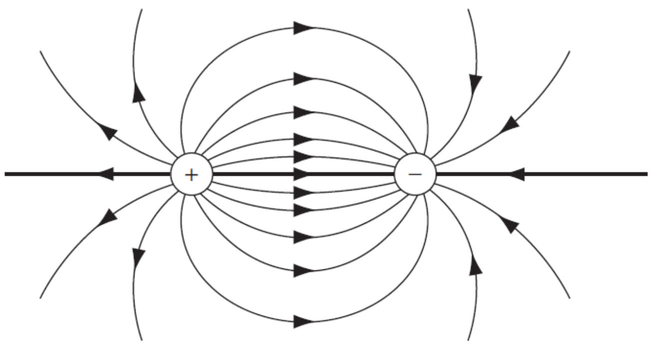
For a positive point charge:
the lines are directed radially outward.



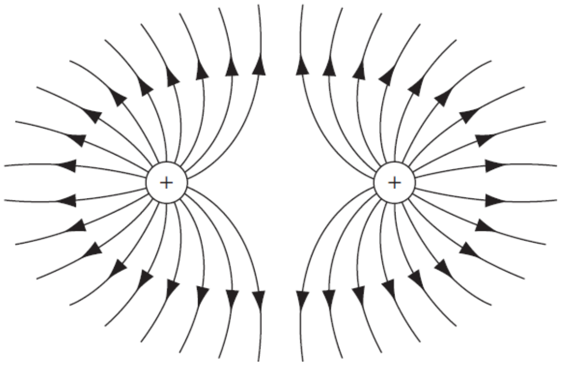
For a negative point charge;
the lines are directed radially inward.



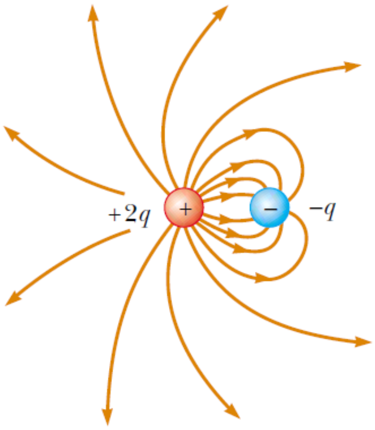
Field lines begin on positive charges and end on negative ones:



Opposite charges

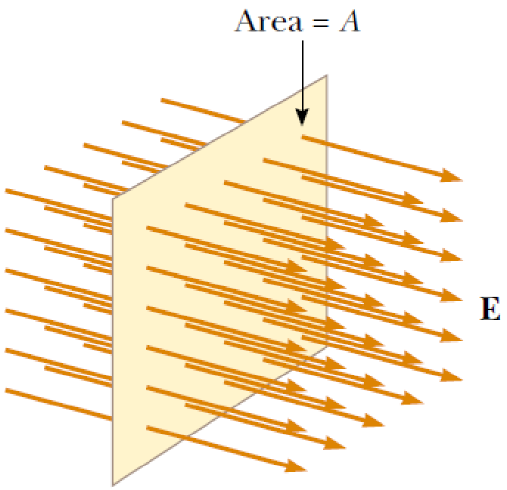


Equal charges



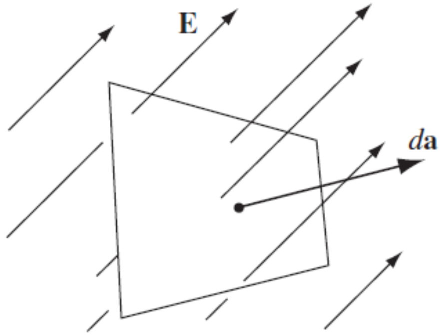
Electric Flux

The *flux* of **E** through a surface *S*,



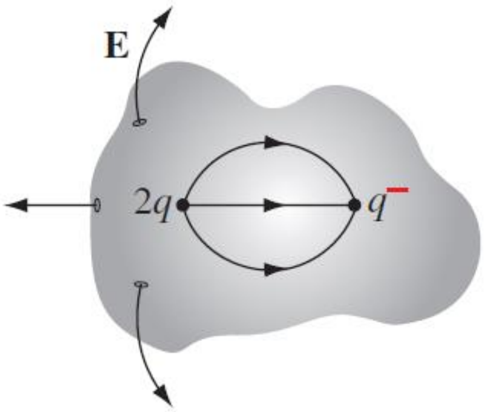
$$\Phi_E \equiv \int_S \mathbf{E} \cdot d\mathbf{a}$$

∴ the flux through any *closed* surface is a measure of the total charge inside.

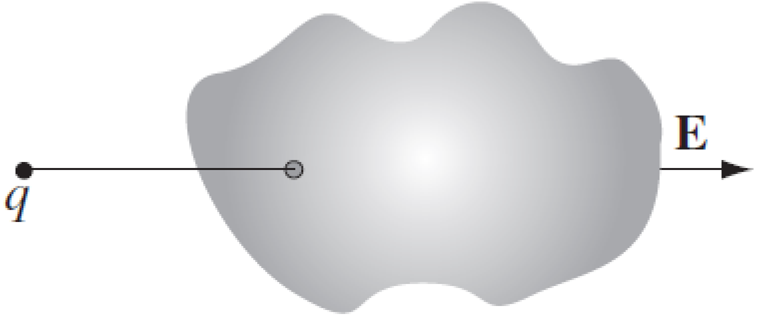


∴ the flux through any *closed* surface is a measure of the total charge inside.

For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside.



A charge *outside* the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other.



This is the *essence* of **Gauss's law**.

E.g.: In the case of a point charge q at the origin, the flux of **E** through a spherical surface of radius r is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0} q$$

Hence, the flux through any surface enclosing the charge is $\boldsymbol{q/\epsilon_0}$**.**

For a bunch of scattered charges; the total field is the (vector) sum of all the individual fields:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i$$

The flux through a surface that encloses them all is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right)$$

For any closed surface;

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

where Q_{enc} is the total charge enclosed within the surface. This is the quantitative statement of ***Gauss's law***.

Gauss's law is an *integral* equation, applying **divergence theorem** to turn it into a *differential* one, by :

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}$$

The *integral* of a *derivative* over a *region* (in this case a *volume*, *V*) is equal to the value of the function at the *boundary* (in this case the *surface S* that bounds the volume).

The divergence theorem:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau$$

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

Rewriting Q_{enc} in terms of the charge density ρ;

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau$$

Gauss's law becomes;

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

since this holds for *any* volume, the integrands must be equal:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho$$

it is the **differential form of Gauss's law.**

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The vector derivatives in spherical coordinates: **Remind!!!**

Gradient:

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl:

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian:

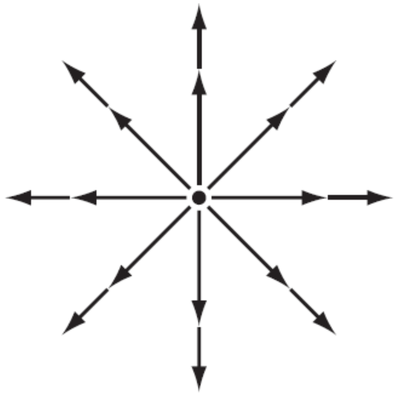
$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

THE DIRAC DELTA FUNCTION

Remind!!!

At every location, **v** is directed radially outward

$$\mathbf{v} = \frac{1}{r^2}\,\hat{\mathbf{r}}$$



in spherical coordinates:

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\int \nabla \cdot \mathbf{v}\, d\tau = 0$$

Suppose we integrate over a sphere of radius *R*, centered at the origin;

$$\oint \mathbf{v} \cdot d\mathbf{a} = \int \left(\frac{1}{R^2}\hat{\mathbf{r}}\right) \cdot (R^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}})$$

$$= \left(\int_0^{\pi} \sin\theta \, d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

Does this mean that the divergence theorem is false?

Remind!!!

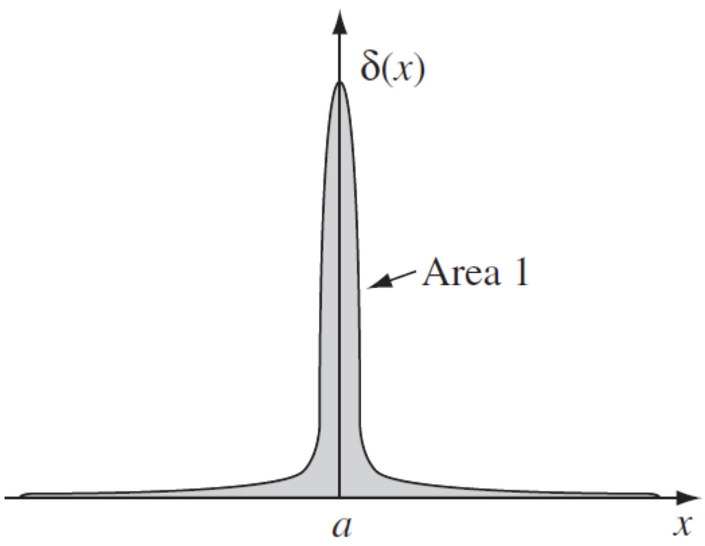
The source of the problem is the point $r = 0$, where $\mathbf{v}$ blows up

It is quite true that $\nabla \cdot \mathbf{v} = 0$ everywhere *except* the origin, but right *at* the origin the situation is more complicated.

The surface integral is *independent of R*; if the divergence theorem is right (and it *is*), we should get

$\int(\nabla \cdot \mathbf{v})\, d\tau = 4\pi$ for *any* sphere centered at the origin, no matter how small.

The One-Dimensional Dirac Delta Function , $\delta(x)$, can be pictured as an infinitely high, infinitesimally narrow "spike," with area 1



$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}$$

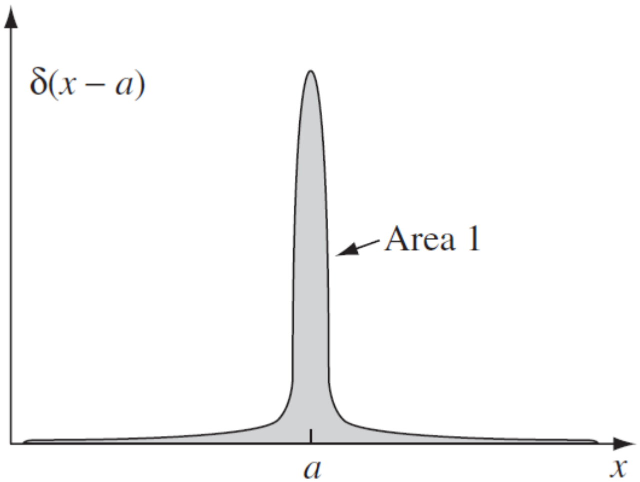
$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

shifting the spike from $x = 0$ to some other point, $x = a$ **Remind!!!**

$$\delta(x-a) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{array} \right\}$$

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$$



The Three-Dimensional Delta Function

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z) \qquad \mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

$$\int_{\text{all space}} \delta^3(\mathbf{r})\, d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz = 1$$

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\, d\tau = f(\mathbf{a})$$

Remind!!!

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r})$$

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

$$\nabla \left(\frac{1}{\mathscr{r}} \right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\nabla^2 \frac{1}{\mathscr{r}} = -4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

The Divergence of E

Remind!!!

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau' \qquad \boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

Gauss's law in differential form;

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

The integral form of Gauss's law:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\tau = \oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}$$