Applications of Gauss's Law

Example 3. Find the field outside a uniformly charged solid sphere of radius R and total charge q.

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \qquad Q_{\text{enc}} = q$$

Figure 18

$\mathbf{E}$ points radially outward, as does $d\mathbf{a}$,

$$\int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{S}} |\mathbf{E}|\, da$$

The *magnitude* of $\mathbf{E}$ is constant over the Gaussian surface;

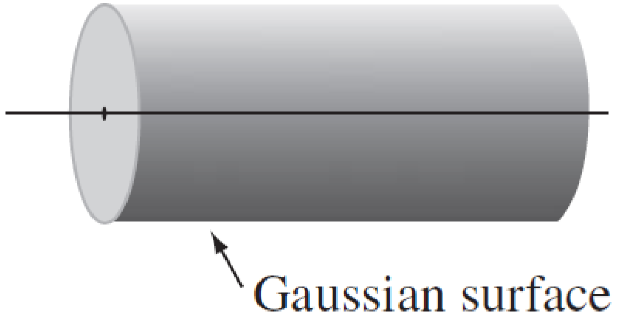
$$\int_{\mathcal{S}} |\mathbf{E}|\, da = |\mathbf{E}| \int_{\mathcal{S}} da = |\mathbf{E}|\, 4\pi r^2$$

$$|\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} q \quad \longrightarrow \quad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

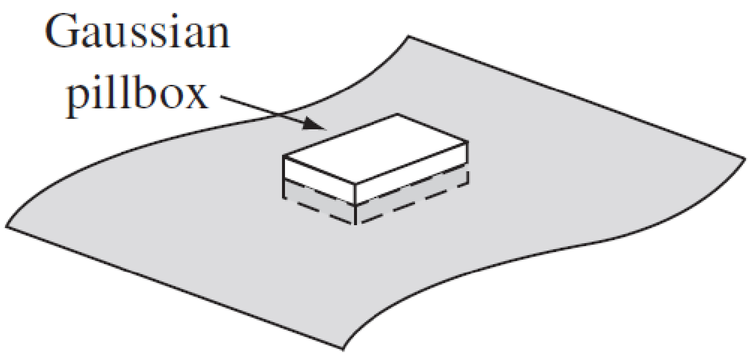
Symmetry is crucial at application of Gauss's law. There are three kinds of symmetry:

1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere.

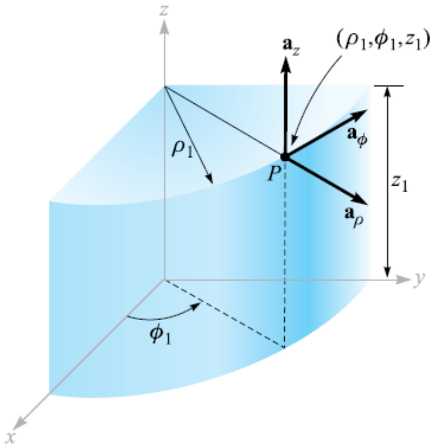
2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder.



3. *Plane symmetry.* Use a Gaussian "pillbox" that straddles the surface.



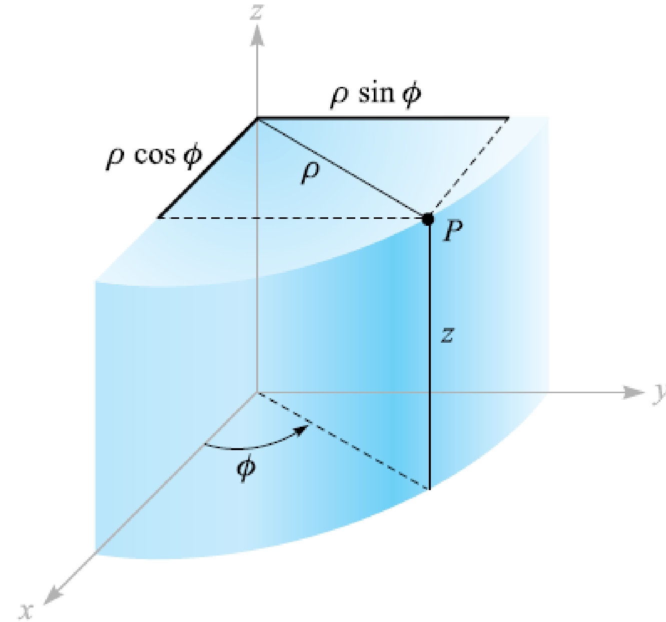
Cylindrical Coordinates



$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$



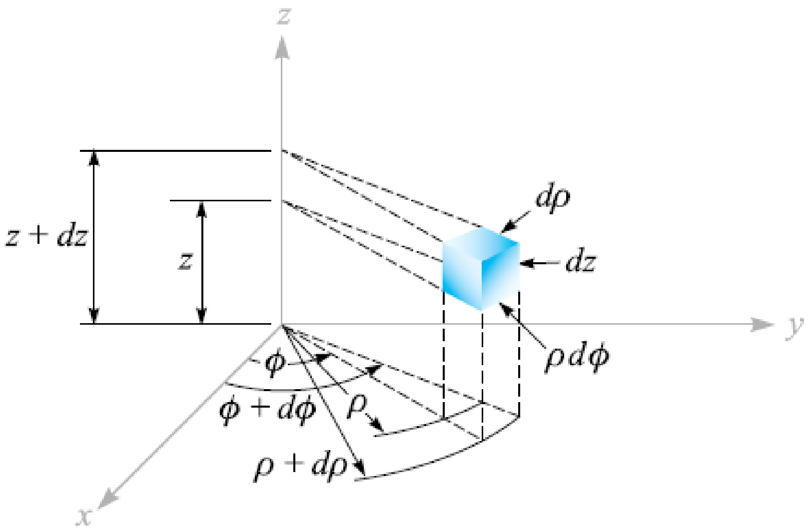
express the cylindrical variables in terms of *x*, *y*, and *z*:

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$

$$\phi = \tan^{-1} \frac{y}{x}$$

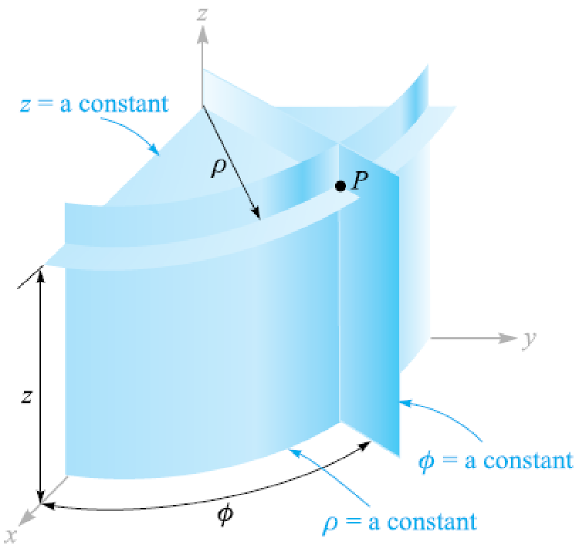
$$z = z$$

A differential volume element in cylindrical coordinates may be obtained by increasing ρ, φ, and z by the differential increments $d\rho$, $d\varphi$, and dz.



Note that $d\rho$ and dz are dimensionally lengths: but $d\varphi$ is not; $\rho d\varphi$ is the length.

The surfaces have areas of
$\rho\, d\rho\, d\varphi$,
$d\rho\, dz$, and
$\rho\, d\varphi\, dz$,
and the volume becomes $\rho\, d\rho\, d\varphi\, dz$.



Example 4. A long cylinder carries a charge density that is proportional to the distance from the axis: $\rho = ks$, for some constant k. Find the electric field inside this cylinder.

Draw a Gaussian cylinder of length l and radius s;

$$\oint_{S} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

Figure 21

$$Q_{\text{enc}} = \int \rho \, d\tau = \int (ks')(s' \, ds' \, d\phi \, dz) = 2\pi k l \int_0^s s'^2 \, ds' = \tfrac{2}{3}\pi k l s^3$$

(integrated φ from 0 to 2π, dz from 0 to l.

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}| \, da = |\mathbf{E}| \int da = |\mathbf{E}| \, 2\pi s l$$

(**E** must point radially outward, so for the curved portion of the Gaussian cylinder adds up, while the two ends contribute nothing - here **E** is perpendicular to $d\mathbf{a}$.)

$$|\mathbf{E}| \, 2\pi s l = \frac{1}{\epsilon_0} \frac{2}{3} \pi k l s^3 \longrightarrow \mathbf{E} = \frac{1}{3\epsilon_0} k s^2 \hat{\mathbf{s}}$$

Example 5. An infinite plane carries a uniform surface charge σ. Find its electric field.

Gaussian pillbox," extending equal distances above and below the plane;

Figure 22

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}}$$

$$\int \mathbf{E} \cdot d\mathbf{a} = 2A|\mathbf{E}|$$

$$Q_{\text{enc}} = \sigma A$$

$$2A\,|\mathbf{E}| = \frac{1}{\epsilon_0} \sigma A \quad \longrightarrow \quad \mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}$$

$\hat{\mathbf{n}}$ is a unit vector pointing away from the surface.

Example 6. Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

Figure 23 - 24

The field between the plates is σ/ϵ_0, and points to the right; elsewhere it is zero.

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