MULTIPOLE EXPANSION

Approximate Potentials at Large Distances

Example. A (physical) **electric dipole** consists of two equal and opposite charges ($\pm q$) separated by a distance d. Find the approximate potential at points far from the dipole.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

Figure 26

From the law of cosines;

$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \cancel{\frac{d^2}{4r^2}}\right)$$

for $r \gg d$,

Then, binomial expansion yields

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right) \Rightarrow \frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

If we put together a pair of equal and opposite *dipoles* to make a **quadrupole**, $1/r^3$; for back-to-back quadrupoles (an octopole), it goes like $1/r^4$;

Figure 27

To develop a systematic expansion for the potential of *any* localized charge distribution, the potential at $\boldsymbol{r}$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$

Figure 28

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

where α is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Then,

$$\mathscr{r} = r\sqrt{1+\epsilon} \qquad \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$$

for $\mathbf{r} \gg \mathbf{r}'$, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

Substitute ϵ,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \overbrace{\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)} + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \ldots\right]$$

Re-arrange for like powers of (***r′/r***);

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \ldots\right]$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'$$ $\Longrightarrow$ the **multipole expansion** of V in powers of $1/r$;

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \ldots\right]$$

The first term is the ***monopole contribution*** (it goes like $1/r$); the second ($n = 1$) is the dipole (it goes like $1/r^2$); …

Like powers of (***r′/r***) coefficients (the terms in parentheses) are Legendre polynomials!

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$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau' \Longrightarrow V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$$

(the **multipole expansion** of V in powers of $1/r$.)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')\,d\tau' + \dots\right]$$

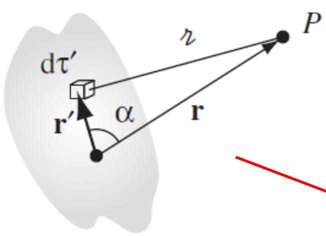
$$\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= (3x^2-1)/2\\
P_3(x) &= (5x^3-3x)/2\\
P_4(x) &= (35x^4-30x^2+3)/8\\
P_5(x) &= (63x^5-70x^3+15x)/8
\end{aligned}$$

The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{\mathrm{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad\qquad Q = \int \rho\, d\tau$$

If the total charge is zero, the dominant term in the potential will be the dipole:



$$V_{\mathrm{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'$$

$$r'\cos\alpha = \mathbf{r}'\cdot\hat{\mathbf{r}} \quad \Longrightarrow \quad V_{\mathrm{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\underbrace{\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'}$$

$$\left(\, r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}' \,\right)$$

dipole moment
of the distribution

$$V_{\mathrm{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \Longleftarrow \quad \mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$

(translates in the usual way for point, line, and surface charges.)

The dipole moment of a collection of *point* charges:

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

E.g.: For a **physical dipole** (equal and opposite charges, ±*q*);

Figure 29

$$\mathbf{p} = q\mathbf{r}'_+ - q\mathbf{r}'_- = q(\mathbf{r}'_+ - \mathbf{r}'_-) = q\mathbf{d}$$

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The Electric Field of a Dipole

Let **p** is at the origin and points in the *z* direction

$$V_{\text{dip}}(r, \theta) = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

Figure 36

Gradient in spherical coordinates:

$$\boldsymbol{\nabla} T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

$$\mathbf{E}_{\text{dip}}(r, \theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

(Notice that the dipole field falls off as the inverse *cube* of *r*)