THE ELECTRIC DISPLACEMENT

Gauss's Law in the Presence of Dielectrics

The field attributable to bound charge plus the field due to **free charge**, ρ_f .

Total charge inside a dielectric is:

$$\rho = \rho_b + \rho_f$$

Bound charges in a dielectric:

$$\rho_b = -\nabla \cdot \mathbf{P} \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

Gauss's law for the electric field is therefore : $\left(\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \right)$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

E is the total field, not just what is generated by the polarization.

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Define: **Electric displacement** →

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

→ Differential form of Gauss's law in presence of a dielectric

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}}$$

→ Integral form of Gauss's law in presence of a dielectric

Q_{fenc} denotes the total free charge enclosed in the volume.

Example 4. A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation out to a radius a. Find the electric displacement.

Figure 17

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

$$\oint D\hat{\mathbf{s}} \cdot da\, \hat{\mathbf{s}} = Q_{f_{enc}} \longrightarrow D(2\pi s L) = \lambda L \longrightarrow \mathbf{D} = \frac{\lambda}{2\pi s}\hat{\mathbf{s}}$$

this formula holds both within the insulation and outside it.

What is the electric field outside the dielectric ($s > a$) ?

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} \qquad (\text{since } \mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \text{ and } \mathbf{P} = 0)$$

What is the electric field inside the dielectric ($s < a$) ?

Since **P** inside is not known, electric field inside cannot be found.

Boundary Conditions in Dielectric

Figure

Boundary Conditions on electric field

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \longleftrightarrow \oint_{surf} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$\Rightarrow \quad \mathrm{E}^{\perp}{}_{\text{above}} - \mathrm{E}^{\perp}{}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = 0 \longleftrightarrow \oint_{path} \mathbf{E} \cdot d\mathbf{l} = \mathbf{0}$$

$$\Rightarrow \quad \mathrm{E}^{||}{}_{\text{above}} - \mathrm{E}^{||}{}_{\text{below}} = 0$$

Boundary Conditions on electric displacement

$$\nabla \cdot \mathbf{D} = \rho_f \longleftrightarrow \oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}} \Rightarrow D^{\perp}{}_{\text{above}} - D^{\perp}{}_{\text{below}} = \sigma_f$$

$$\nabla \times \mathbf{D} = \epsilon_0 (\nabla \times \mathbf{E}) + \nabla \times \mathbf{P} = \nabla \times \mathbf{P} \Rightarrow D^{||}{}_{\text{above}} - D^{||}{}_{\text{below}} = P^{||}{}_{\text{above}} - P^{||}{}_{\text{below}}$$

LINEAR DIELECTRICS

Two ways that an atom/molecule acquires ***dipole moment***:

Stretch of an atom/molecule

Figure

Figure

Rotation of a polar atom/mol

$\Rightarrow$ $\mathbf{p} = \alpha\, \mathbf{E}_{\text{else}}$

- α is the atomic polarizability
- $\mathbf{E}_{\text{else}}$ is the external field, caused by everything except the atom

Dipole moment **p** is a microscopic quantity. However, polarization **P** (dipole moment per unit volume) is a macroscopic quantity.

If $\mathbf{E}$ is not too strong

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

- χ_e is called the electrical susceptibility and depends on the details (microscopic and macroscopic of the medium)
- $\mathbf{E}$ is the total field caused by everything.
- How are $\boldsymbol{\alpha}$ and χ_e connected?

Linear Dielectrics

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

- Medium that obeys this equation is called the linear dielectric
- When polarization is proportional to the square or higher-order terms in $\mathbf{E}$, the medium is called the nonlinear dielectric.
- At strong enough $\mathbf{E}$ every medium becomes nonlinear.
- In general χ_e is a tensor and is called the susceptibility tensor.

The electric displacement **D** for a linear dielectric

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad = \epsilon_0\, \mathbf{E} + \epsilon_0 \chi_e\, \mathbf{E} \quad = \epsilon_0\,(1+\chi_e)\mathbf{E}$$

Define: $\epsilon \equiv \epsilon_0(1+\chi_e)$

Define: $\epsilon_r \equiv \dfrac{\epsilon}{\epsilon_0} = (1+\chi_e)$

$$\mathbf{D} = \epsilon \mathbf{E}$$

- The electric displacement **D** is also proportional to **E**
- ϵ is called the permittivity of the material.
- ϵ_r is called the relative permittivity or the dielectric constant.
- $n = \sqrt{\epsilon_r}$ is called the refractive index.

How are $\boldsymbol{\alpha}$ and $\boldsymbol{\chi_e}$ connected? ----- Without derivation;

$$\alpha = \frac{3\epsilon_0}{N}\frac{(\epsilon_r - 1)}{(\epsilon_r + 2)}$$

- Clausius-Mossotti formula
- Connects a microscopic quantity to a macroscopic quantity

Data from Handbook of Chemistry and Physics, 91st ed. (Boca Raton: CRC Press, 2010).

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

Example 5. A metal sphere of radius *a* carries a charge *Q*. It is surrounded, out to radius *b*, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{0}$.

Using Gauss Law for r > a :

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \quad \text{for all points } r > a.$$

Figure 20

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } a < r < b, \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > b. \end{cases}$$

The potential at the center, r = 0

$$V = -\int_{\infty}^{0} \mathbf{E} \cdot d\mathbf{l} = -\int_{\infty}^{b} \left(\frac{Q}{4\pi\epsilon_0 r^2} \right) dr - \int_{b}^{a} \left(\frac{Q}{4\pi\epsilon r^2} \right) dr - \int_{a}^{0} (0)\, dr$$

$$= \frac{Q}{4\pi} \left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b} \right).$$

The polarization and the bound charges:

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2} \hat{\mathbf{r}}$$

$$\rho_b = -\nabla \cdot \mathbf{P} = 0 \quad \longleftarrow \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}.$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}, & \text{at the outer surface,} \\[2ex] \dfrac{-\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}, & \text{at the inner surface.} \end{cases}$$