Convolution

Lecture 9

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Recap:

We have seen systems with:

- memory
- invertibility
- causality
- stability
- linearity
- time invariance

and properties of LTI Systems in the last lecture.

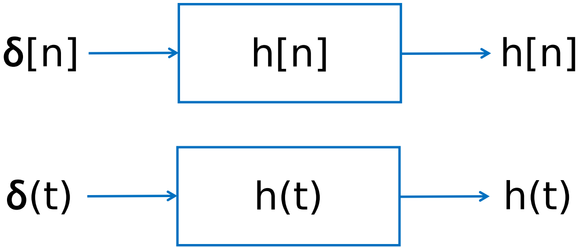
In this lecture, we will learn about convolution operation.

Convolution:

Definition: It is a mathematical way of combining two signals to form a third signal and commonly denoted with the operator $*$.

Convolution Theorem: It is one of the most important theorems for LTI systems. Convolution theorem states that the response of a system at zero initial conditions due to any input is the convolution of that input and the system's impulse response.

Impulse Response of a System



Impulse response of a system is the system's output when its input is an impulse.

Impulse Response:

Impulse response is usually denoted with $h[n]$ for discrete time systems (DTS) and $h(t)$ for continuous time systems (CTS).

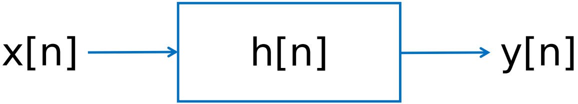
The output of an LTI system to any input $x[n]$ for DTS and $x(t)$ for CTS can be found by taking the convolution of that input signal with the impulse response:

$$y[n] = x[n] * h[n]$$
$$y(t) = x(t) * h(t)$$

Discrete Time Convolution:

Convolution of two discrete time (DT) signals, $x_1[n]$ and $x_2[n]$, is calculated as:

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

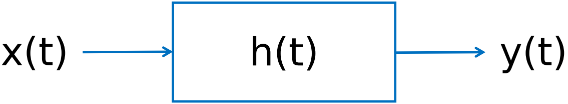


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Continuous Time Convolution:

Convolution of two continuous time (CT) signals, $x_1(t)$ and $x_2(t)$, is calculated as:

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) x_2(t-\tau) d\tau$$



$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Commutative Property:

Commutative property states that the order in which two signal are convolved does not change the result of convolution.

$$x_1[n] * x_2[n] = x_2[n] * x_1[n]$$
$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Associative Property:

The convolution operation satisfies associative property, which is:

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$
$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

Remember cascaded LTI systems from the last lecture.

Distributive Property

The convolution operation satisfies the distributive property, which is:

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$
$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$

Remember parallel LTI systems from the last lecture.

Convolution with an Impulse:

If a signal is convolved with an impulse ($\delta[n]$), the result is the exact replica of the input:

$$x[n] * \delta[n] = x[n]$$
$$x(t) * \delta(t) = x(t)$$

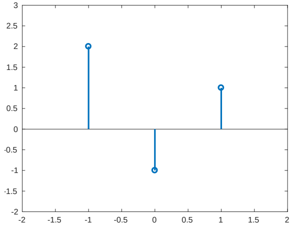
Convolution with a Time-Shifted Impulse:

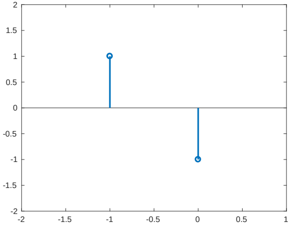
If a signal is convolved with a time-shifted impulse ($\delta[n - n_0]$), the result is time shifted version of the input with the same amount as the impulse:

$$x[n] * \delta[n - n_0] = x[n - n_0]$$
$$x(t) * \delta(t - \tau) = x(t - \tau)$$

Convolution Example:

Let x[n] and h[n] be:





Hence:

$$x[n] = 2\delta[n+1] - \delta[n] + \delta[n-1]$$
$$h[n] = \delta[n+1] - \delta[n]$$

Convolution Example Solution:

$$
\begin{aligned}
x[n] * h[n] &= x[n] * (\delta[n+1] - \delta[n]) \\
&= x[n] * \delta[n+1] - x[n] * \delta[n] \\
&= 2\delta[n+2] - \delta[n+1] + \delta[n] - 2\delta[n+1] + \delta[n] - \delta[n-1] \\
&= 2\delta[n+2] - 3\delta[n+1] + 2\delta[n] - \delta[n-1]
\end{aligned}
$$

Example Solution in Matlab

The code is:

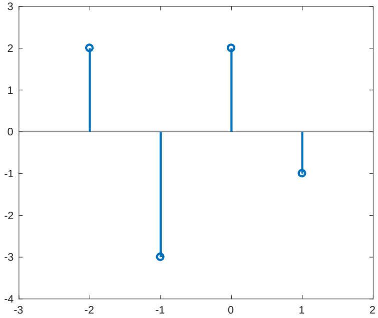
```
x = [2, -1, 1];
h = [1, -1];
y = conv(x,h);
disp(y)
```

The output is:

```
>> conv_exp
     2    -3     2    -1
```

Example Solution Plot:

If we plot the output:



Convolution Example on Matrices:

Images are stored as matrices in environments like MATLAB.

2D and 3D convolutions are very common in image processing.

Here is an example:

https://upload.wikimedia.org/wikipedia/commons/4/4f/3D_Convolution_Animation.gif

1D CT Convolution Example:

https://upload.wikimedia.org/wikipedia/commons/a/a8/Splot1.gif

Summary:

In this lecture, we learned the following topics:

- Impulse response of an LTI system (Recap)
- DT and CT convolution
- Properties of convolution
- Convolution with Time-Shifted impulse
- Convolution in Matlab

Next Lecture

In the next lecture, we will solve examples for both DT and CT convolution.
Please read related chapters from Oppenheim's book before the next lecture.