# Discrete-Time Fourier Series

## Lecture 11

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**Content of This Lecture**:

In this lecture, we will learn how to represent discrete time periodic signals with complex exponentials.

Later, we will learn how to extend this to aperiodic signals (Discrete Time Fourier Transform).

## Response of LTI to Complex Exponentials:

If we can represent a signal as a linear combination of basic signals then the response of an LTI system to this signal will be the linear combination of responses to each of these basic signals.

For LTI systems these basic signals are complex exponentials (for continuous time: $e^{st}$, for discrete time: $z^n$). Because:

- Complex exponentials exit an LTI system with only a change (if any) in their amplitude.
- Broad classes of signals can be represented with linear combination of complex exponentials.

## Eigenfunctions of LTI Systems:

By definition, an eigenfunction of a system is a signal such that the response of the system to that signal is just the signal itself multiplied with a constant.

Complex exponentials are eigenfunctions of LTI systems.

They enter an LTI system and leave the system as they are but just multiplied by a constant.

## Periodic Signals:

Remember, if signal $x[n]$ is periodic with period $N_0$, then $x[n] = x[n + N_0]$.

- Fundamental period: $N_0$ where $N = kN_0$ and $k \in Z$ (sec).
- Fundamental frequency: $\Omega_0 = 2\pi f_0$ (radians/sec).
- Fundamental period: $N_0 = \frac{2\pi}{\Omega_0}$

## Harmonically-Related Complex Exponentials:

$$\sigma_k[n] = e^{jk2\frac{2\pi}{N}n}, k \in Z \quad (1)$$

where $N$ is the period.

There are only $N$ distinct $e^{jk\frac{2\pi}{N}n}$.

$e^{jk\frac{2\pi}{N}n} = e^{j(k+lN)\frac{2\pi}{N}n}$ where $l \in Z$

## Representing A Periodic Signal with Complex Exponentials:

We can represent a discrete time periodic signal $x[n]$ with harmonically related complex exponentials as:

$$x[n] = \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}n} \quad (2)$$

This representation is called Fourier Series representation of $x[n]$ which is a continuous time periodic signal with a fundamental period of $N_0$.

**Important Remarks**:

$$x[n] = \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}n} \tag{3}$$

- In this representation, $a_k$'s are called Fourier series coefficients (or spectral coefficients)
- Almost all periodic discrete time signals that are common in engineering can be written in this form.

## Finding Fourier Series Coefficients ($a_k$'s):

We can calculate $a_k$'s by evaluating the following summation:

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \tag{4}$$

The boundary encapsulates any period of the signal.

## Properties of Fourier Series Coefficients - 1:

If a DT periodic signal satisfies:

$$x[n] = x^*[n] \tag{5}$$

Hence $x(t)$ is a real CT periodic signal. Then,

$$a_k = a^*_{-k} \tag{6}$$

## Properties of Fourier Series Coefficients - 2:

If $x[n]$ is real and even then $a_k$ is also real and even:

$$x[n] = x[n] \Rightarrow a_k = a_{-k} \tag{7}$$

If $x[n]$ is real and odd then $a_k$ is purely imaginary and odd:

$$x[n] = -x[n] \Rightarrow a_k = -a_{-k} \tag{8}$$

## Linear Combination of Two CT Periodic Signals:

$$x[n] = \sum_{k=k_0}^{k_0+N-1} a_k e^{jk\frac{2\pi}{N}n}, \tag{9}$$

$$y[n] = \sum_{k=k_0}^{k_0+N-1} b_k e^{jk\frac{2\pi}{N}n}, \tag{10}$$

$$Ax[n] + By[n] = \sum_{k=k_0}^{k_0+N-1} (Aa_k + Bb_k)e^{jk\frac{2\pi}{N}n} \tag{11}$$