CHAPTER 3. APPLICATIONS of FIRST ORDER DIFFERENTIAL EQUATIONS

3.3. Growth and Decay Problems

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that $\frac{dN}{dt}$ the time rate of change of this amount of substance is proportional to the amount of substance present, then

$$\frac{dN}{dt} = kN$$

or

$$\frac{dN}{dt} - kN = 0,$$

where k is the constant of proportionally

Example 1) The rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. Half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years.

(a) What percentage of the original radioactive nuclei will remain after 4500 years?

(b) In how many years will only $one - tenth$ of the original number remain?

Solution. Let N be the amount of radioactive nuclei present after t years. Since the nuclei decay at a rate proportional to the amount of present, we have

$$\frac{dN}{dt} = -kN, \tag{1}$$

where $k > 0$ is a constant of proportionality. Letting N_0 denote the amount initially present, we have initial condition

$$N(0) = N_0.$$

Moreover, since half of the original number of radioactive nuclei have undergone disintegration in a period of 1500 years, we also have

$$N(1500) = \frac{1}{2}N_0.$$

The differential equation (1) is clearly seperable, integrating it we have

$$\ln |N| = -kt + \ln c$$

or

$$N(t) = ce^{-kt}.$$

Applying the initial condition $N(0) = N_0$, we get $N_0 = c$ and hence we have

$$N(t) = N_0 e^{-kt}. \tag{2}$$

To determine k apply the condition $N(1500) = \frac{1}{2}N_0$. So, we have $e^{-k} = \left(\frac{1}{2}\right)^{1/1500}$. Substituting e^{-k} into (2), we obtain

$$N(t) = N_0 \left(\frac{1}{2}\right)^{t/1500}. \tag{3}$$

(a) $N(4500) = N_0 \left(\frac{1}{2}\right)^3$. So, one-eight or 12.5 % of the original number remain after 4500 years.

(b) From (3)

$$\frac{1}{10}N_0 = N_0 \left(\frac{1}{2}\right)^{t/1500}$$

or

$$\frac{1}{10} = \left(\frac{1}{2}\right)^{t/1500}.$$

Using logarithms, we obtain

$$\ln \frac{1}{10} = \frac{t}{1500} \ln \left(\frac{1}{2}\right).$$

From this it follows that

$$t = 1500 \frac{\ln\left(\frac{1}{10}\right)}{\ln\left(\frac{1}{2}\right)} = 1500 \frac{\ln 10}{\ln 2}.$$

3.4. Temperature Problems

Newton's law of cooling, which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surronding medium. Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is dT/dt, and Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -k\left(T - T_m\right)$$

or

$$\frac{dT}{dt} + kT = kT_m,$$

where k is a positive constant of proportionally.

Example 1) A body of temperature 80 $^\circ F$ is placed at time $t = 0$ in a medium the temperature of which is maintained at 50 $^\circ F$. At the end of 5 minutes, the body has cooled to a temperature of 70 $^\circ F$.

(a) What is the temperature of the body at the time of 10 minutes?

(b) When will the temperature of the body be 60 $^\circ F$?

Solution. Since $T_m = 50$ we have following seperable differential equation

$$\frac{dT}{dt} + kT = 50k$$

or

$$\frac{dT}{kT - 50k} = -dt. \tag{4}$$

Integrating (4), we get

$$\frac{1}{k} \ln (kT - 50k) = -t + \ln c_1$$

or

$$T(t) = 50 + ce^{-kt}.$$

Applying the initial condition $T(0) = 80$ we get $c = 30$ and so

$$T(t) = 50 + 30e^{-kt}.$$

From the other condition $T(5) = 70$, we obtain that

$$e^{-k} = \left(\frac{2}{3}\right)^{1/5}$$

Substituting e^{-k} into (4), we obtain

$$T(t) = 50 + 30\left(\frac{2}{3}\right)^{t/5}. \tag{5}$$

(a) $T(10) = 50 + 30\left(\frac{2}{3}\right)^2 = \frac{190}{3}$.

(b) It is asked that for which t the relation $T(t) = 60$ is satisfied. From (5) we have

$$60 = 50 + 30\left(\frac{2}{3}\right)^{t/5}$$

or

$$\left(\frac{1}{3}\right) = \left(\frac{2}{3}\right)^{t/5}.$$

Using logarithms, we obtain

$$t = 5\frac{\ln \frac{1}{3}}{\ln \frac{2}{3}}.$$