Calculus II
Week 8 Lecture

Oktay Olmez and Serhan Varma

Sequences

A sequence is a set of ordered numbers. For example, the sequence 2, 4, 6, 8, ... has 2 as its first term, 4 as its second, etc. The n-th term in a sequence is usually called a_n. The terms of a sequence may be arbitrary, or they may be defined by a formula, such as $a_n = 2n$.

$$a_1 - \text{first term}$$
$$a_2 - \text{second term}$$
$$\vdots$$
$$a_n - n^{th} \text{ term}$$

Sequences

We denote a sequence by the following notations:

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

We can also treat the sequence terms as function evaluations :

$$f(n) = a_n = \frac{n+1}{n}$$

Sequences

We say that

$$\lim_{n\to\infty} a_n = L$$

if we can make a_n as close as L as we want for all sufficiently large n.

$$\lim_{n\to\infty} a_n = \infty$$

if we can make a_n as large as we want for all sufficiently large n.

$$\lim_{n\to\infty} a_n = -\infty$$

if we can make a_n as small as we want for all sufficiently large n.

Theorem

Given the sequence $\{a_n\}$ *if we have a funnction* $f(x)$ *such that* $f(n) = a_n$ *and* $\lim_{x\to\infty} f(x) = L$ *then*

$$\lim_{n\to\infty} a_n = L$$

Sequences: Limit properties

If $\{a_n\}$ and $\{b_n\}$ convergent sequences then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
2. $\lim\limits_{n\to\infty} c a_n = c \lim\limits_{n\to\infty} a_n$
3. $\lim\limits_{n\to\infty} (a_n\, b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$
4. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided $\lim\limits_{n\to\infty} b_n \neq 0$

Sequences: Squeeze theorem

Theorem

If $a_n \leq c_n \leq b_n$ *for all* $n > N$ *for some* N *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ *then*

$$\lim_{n\to\infty} c_n = L$$

Sequences: Limits

Theorem

If $\lim_{n\to\infty} |a_n| = 0$ *then*

$$\lim_{n\to\infty} a_n = 0$$

Theorem

The sequence $\{r^n\}_{n=0}^{\infty}$ *converges if* $-1 < r < 1$ *and diverges otherwise. If it converges then*

$$\lim_{n\to\infty} r^n = 0$$

Sequences: bounded sequence

Given any sequence $\{an\}$we have the following.
We call the sequence increasing if $a_n < a_{n+1}$ for every n.
We call the sequence decreasing if $a_n > a_{n+1}$ for every n.
If $\{an\}$ is an increasing sequence or a decreasing sequence then we call it monotonic.
If there exists a number m such that $m \leq a_n$ for every n we say the sequence is bounded below.
If there exists a number Msuch that $a_n \leq M$ for every n we say the sequence is bounded above.
If the sequence is both bounded below and bounded above we call the sequence bounded.

Sequences: bounded sequences

Theorem

If $\{an\}$ *is bounded and monotonic then* $\{an\}$ *is convergent.*