Calculus II
Week 12 Lecture

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Taylor Series

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots
\end{aligned}
$$

Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Taylor Series

n-th degree Taylor polynomial of $f(x)$

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Taylor Series

n-th degree Taylor polynomial of $f(x)$ is just the partial sum for the series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Remainder:

$$R_n(x) = f(x) - T_n(x)$$

Taylor Series

Suppose $f(x) = T_n(x) + R_n(x)$. Then if

$$\lim_{n \to \infty} R_n(x) = 0$$

for $|x - a| < R$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Series

Find the Taylor Series for $f(x) = e^x$ about $x = 0$.
Find the Taylor Series for $f(x) = e^{-x}$ about $x = 0$.
Find the Taylor Series for $f(x) = e^{-x}$ about $x = -2$.
Find the Taylor Series for $f(x) = \sin x$ about $x = 0$.
Find the Taylor Series for $f(x) = \ln x$ about $x = 2$.

Taylor Series

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Taylor Series

Determine a Taylor Series about $x = 0$ for the following integral

$$\int \frac{\sin x}{x}\, dx$$

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

Taylor Series

n-th degree polynomial:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Find T_2, T_4 and T_8 for $f(x) = \cos x$ about $x = 0$

$$T_2(x) = 1 - \frac{x^2}{2}$$
$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
$$T_8(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}$$

Binomial Series

$$
\begin{aligned}
(a+b)^n &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i}\, b^i \\
&= a^n + na^{n-1}b + \frac{n\,(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n
\end{aligned}
$$

Binomial Series

$$
\begin{aligned}
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\
&= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots
\end{aligned}
$$