• Roots of Equations, Bracketing Methods, Open Methods [1-6]

References:

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$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

$$f(x_l)f(x_r) < 0 \qquad x_r = x_u$$

$$f(x_l)f(x_r) > 0 \qquad x_r = x_l$$

$$f(x_l)f(x_r) = 0 \qquad x_r = x_r$$

$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

$$\varepsilon_t = \left| \frac{x_{true} - x_r^{new}}{x_{true}} \right| 100\%$$

The false position method:

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

First iteration:

$$f(c) = \frac{667.38}{c}(1 - e^{-0.146843\,c}) - 40$$

$$x_l = 12 \qquad f(x_l) = 6.0699$$

$$x_u = 16 \qquad f(x_u) = -2.2688$$

$$x_r = 16 - \frac{-2.2688(12 - 16)}{6.0699 - f(-2.2688)} = 14.9113$$

$$f(x_r) = f(14.9113) = \frac{667.38}{c}(1 - e^{-0.146843\,c}) - 40 = -0.2543$$

$$f(x_l)f(x_r) = f(12)f(14.9113) = 6.0699 * (-0.2543) < 0$$

So; $x_r = x_u$

Second iteration:

$$x_l = 12 \qquad f(x_l) = 6.0699$$

$$x_u = 14.9113 \qquad f(x_u) = -0.2543$$

$$x_r = 14.9113 - \frac{-0.2543(12 - 14.9113)}{6.0699 - (-0.2543)} = 14.7942$$

$$\varepsilon_a = \left|\frac{x_r^{new} - x_r^{old}}{x_r^{new}}\right| 100\%$$

$$\varepsilon_a = \left|\frac{14.7942 - 14.9113}{14.7942}\right| 100\% = 0.7915$$

$$\varepsilon_t = \left|\frac{x_{true} - x_r^{new}}{x_{true}}\right| 100\%$$

$$\varepsilon_t = \left|\frac{14.7802 - 14.7942}{14.7802}\right| 100\% = 0.0947 < 0.5$$

$$\boldsymbol{x_{i+1}} = \boldsymbol{x_i} - \frac{f(x_i)}{\boldsymbol{f}'(x_i)}$$

$$\varepsilon_a = \left|\frac{{x_r}^{new} - {x_r}^{old}}{{x_r}^{new}}\right| 100\%$$

The Newton-Raphson method:

$$\boldsymbol{x_{i+1}} = \boldsymbol{x_i} - \frac{f(x_i)}{\boldsymbol{f}'(x_i)}$$

$$f(x) = e^{-x} - x$$

The first derivative of the function is:

$$\boldsymbol{f}'(x) = -e^{-x} - 1$$

Which can be substituted along with the original function into the Newton-Raphson equation:

$$\boldsymbol{x_{i+1}} = \boldsymbol{x_i} - \frac{e^{-x} - x}{-e^{-x} - 1}$$

Starting with the guess of $x_0 = 0.5$

First iteration:

$$\boldsymbol{x_{i+1}} = \mathbf{0.5} - \frac{e^{-0.5} - 0.5}{-e^{-0.5} - 1} = \mathbf{0.5663}$$

$$\varepsilon_a = \left|\frac{{x_r}^{new} - {x_r}^{old}}{{x_r}^{new}}\right| 100\%$$

$$\varepsilon_a = \left|\frac{0.5663 - 0.5}{0.5663}\right| 100\% = 11.7$$

Second iteration:

$x_i = 0.5663$

$$\boldsymbol{x_{i+1}} = \mathbf{0.5663} - \frac{e^{-0.5663} - 0.5663}{-e^{-0.5663} - 1} = \mathbf{0.5671}$$

$$\varepsilon_a = \left|\frac{0.5671 - 0.5663}{0.5671}\right| 100\% = 0.14$$

So;

x = 0.5671

Inverse Quadratic Interpolation method:

$$x_{i+1} = \frac{y_{i-1}y_i}{(y_{i-2}-y_{i-1})(y_{i-2}-y_i)}x_{i-2} + \frac{y_{i-2}y_i}{(y_{i-1}-y_{i-2})(y_{i-1}-y_i)}x_{i-1} + \frac{y_{i-2}y_{i-1}}{(y_i-y_{i-2})(y_i-y_{i-1})}x_i$$

$$\varepsilon_t = \left|\frac{x_{true} - x_r^{new}}{x_{true}}\right| 100\%$$

First iteration:

$$y = f(x) = e^{-x} - x = 0$$

$x_{i-2} = 0.1 \qquad y_{i-2} = f(0.1) = e^{-0.1} - 0.1 = 0.8048$

$x_{i-1} = 0.5 \qquad y_{i-1} = f(0.5) = e^{-0.5} - 0.5 = 0.1065$

$x_i = 1.0 \qquad y_i = f(1.0) = e^{-1.0} - 1.0 = -0.6321$

$$x_{i+1} = \frac{y_{i-1}y_i}{(y_{i-2}-y_{i-1})(y_{i-2}-y_i)}x_{i-2} + \frac{y_{i-2}y_i}{(y_{i-1}-y_{i-2})(y_{i-1}-y_i)}x_{i-1} + \frac{y_{i-2}y_{i-1}}{(y_i-y_{i-2})(y_i-y_{i-1})}x_i$$

$$x_{i+1} = \frac{0.1065(-0.6321)}{(0.8048-0.1065)(0.8048--0.6321)}0.1 + \frac{0.8048(-0.6321)}{(0.1065-0.8048)(0.1065--0.6321)}0.5$$

$$+\frac{0.8048(0.1065)}{(-0.6321-0.8048)(-0.6321-0.1065)}1.0$$

$$x_{i+1} = -0.0067 + 0.4931 + 0.0807 = 0.5671$$

$$y_{i+1} = f(0.5671) = e^{-0.5671} - 0.5671 = 0.000068 \approx 0$$

$$\boldsymbol{\varepsilon_t = \left|\frac{x_{true} - x_r^{new}}{x_{true}}\right| 100\% = \varepsilon_t = \left|\frac{0.56714 - 0.5671}{0.56714}\right| 100\% = 0.007\,\%}$$

bisection to determine the root of function ;

$$f(x) = e^{-3x} - 2x$$

the initial guesses $x_L = 0.23$ and $x_U = 0.26$

$$x_r = \frac{0.23 + 0.26}{2} = 0.2450$$

Computing the product of the function value at the lower bound and at the midpoint:

$$f(0.23) \times f(0.2450) = (0.0416) * (-0.0105) = -0.0004$$

Which is lower than zero and hence a sign change occurs between the lower bound and the midpoint. Therefore the root is between 0.23 and 0.2450. The upper bound is redefined as 0.2450 and the root estimate for the second iteration is calculated as;

$$x_L = 0.23, \qquad x_r = x_U = 0.2450$$

$$x_r = \frac{0.2300 + 0.2450}{2} = 0.2375$$

$$\varepsilon_a = \left|\frac{0.2375 - 0.2450}{0.2375}\right| 100\% = 3.16$$

$$f(0.23) * f(0.2375) = (0.0416) * (0.0154) = 0.0006 > 0$$

$$x_r = x_L = 0.2375, \qquad x_U = 0.2450$$

$$x_r = \frac{0.2375 + 0.2450}{2} = 0.2412$$

$$\varepsilon_a = \left|\frac{0.2412 - 0.2375}{0.2412}\right| 100\% = 1.53$$

Iteration	x_L	x_U	x_r	ε_a (%)
1	0.2300	0.2600	0.2450	
2	0.2300	0.2450	0.2375	3.16
3	0.2375	0.2450	0.2412	1.53

x = 0.2412

the multiple-equation Newton-Raphson method to calculate the roots of equations

a correct pair of roots are x= -0.1868 and y=0.5283.

guesses of $x_0 = -0.1600$ and $y_0 = 0.5000$

$$u(x, y) = -x^2 + x - y + 0.75$$
$$v(x, y) = x^2 - y - 5xy$$

$$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{dy} - vi \frac{\partial u_i}{dy}}{\frac{\partial u_i}{dx}\frac{\partial v_i}{dy} - \frac{\partial u_i}{dy}\frac{\partial v_i}{dx}} \qquad y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{dx} - ui \frac{\partial v_i}{dx}}{\frac{\partial u_i}{dx}\frac{\partial v_i}{dy} - \frac{\partial u_i}{dy}\frac{\partial v_i}{dx}}$$

$$u(x, y) = x^2 - x + y - 0.75 = 0 \qquad v(x, y) = y + 5xy - x^2 = 0$$

$$\frac{\partial u_i}{dy} = 1 \qquad \frac{\partial u_i}{dx} = 2x_i - 1$$
$$\frac{\partial v_i}{dy} = 1 + 5x_i \qquad \frac{\partial v_i}{dx} = 5y_i - 2x_i$$

1^{st} iteration

$$\frac{\partial u_0}{dy} = 1 \qquad \frac{\partial u_0}{dx} = 2x_0 - 1 = 2\times(-0.16) - 1 = -1.32$$

$$\frac{\partial v_0}{dy} = 1 + 5x_0 = 1 + 5\times(-0.16) = 0.2 \qquad \frac{\partial v_0}{dx} = 5y_0 - 2x_0 = 5\times 0(.5) - 2\times(-0.16) = 2.82$$

$$J = \begin{bmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{bmatrix} = \begin{bmatrix} -1.32 & 1 \\ 2.82 & 0.2 \end{bmatrix}$$

$$|J| = \frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} - \frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial x} = -1.32\times 0.2 - 1\times 2.82 = -3.084$$

$$u(x_0, y_0) = x_0^2 - x_0 + y_0 - 0.75 = (-0.16)^2 - (-0.16) + 0.50 - 0.75 = -0.0644$$

$$v(x_0, y_0) = y_0 + 5x_0 y_0 - x_0^2 = 0.50 + 5*(-0.16)*(0.50) - (-0.16)^2 = 0.0744$$

If these values substitute into the following equations x_1 and y_1 are solved.

$$x_{i+1} = x_i - \frac{u_i \frac{\partial v_i}{dy} - vi \frac{\partial u_i}{dy}}{\frac{\partial u_i}{dx}\frac{\partial v_i}{dy} - \frac{\partial u_i}{dy}\frac{\partial v_i}{dx}} \qquad y_{i+1} = y_i - \frac{v_i \frac{\partial u_i}{dx} - ui \frac{\partial v_i}{dx}}{\frac{\partial u_i}{dx}\frac{\partial v_i}{dy} - \frac{\partial u_i}{dy}\frac{\partial v_i}{dx}}$$

$$x_1 = x_0 - \frac{u_0 \frac{\partial v_0}{dy} - v_0 \frac{\partial u_0}{dy}}{\frac{\partial u_0}{dx}\frac{\partial v_0}{dy} - \frac{\partial u_0}{dy}\frac{\partial v_0}{dx}} = (-0.16) - \frac{(-0.0644)*(0.2) - (0.0744)*(1)}{-3.084} = -0.1883$$

$$y_1 = y_0 - \frac{v_0 \frac{\partial u_0}{dx} - u_0 \frac{\partial v_0}{dx}}{\frac{\partial u_0}{dx}\frac{\partial v_0}{dy} - \frac{\partial u_0}{dy}\frac{\partial v_0}{dx}} = 0.50 - \frac{(0.0744)*(-1.32) - (-0.0644)*(2.82)}{-3.084} = 0.5270$$

2nd iteration

$$\frac{\partial u_1}{dy} = 1 \qquad \frac{\partial u_1}{dx} = 2x_1 - 1 = 2\times(-0.1883) - 1 = -1.3766$$

$$\frac{\partial v_1}{dy} = 1 + 5x_1 = 1 + 5\times(-0.1883) = 0.0585 \qquad \frac{\partial v_1}{dx} = 5y_1 - 2x_1 = 5\times 0.5270 - 2\times(-0.1883) = 3.0116$$

$$J = \begin{bmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{bmatrix} = \begin{bmatrix} -1.3766 & 1 \\ 3.0116 & 0.0585 \end{bmatrix}$$

$$u(x_1, y_1) = x_1^2 - x_1 + y_1 - 0.75 = (-0.1883)^2 - (-0.1883) + 0.5270 - 0.75 = 7.5689 * 10^{-4}$$

$$v(x_1, y_1) = y_1 + 5x_1y_1 - x_1^2 = 0.5270 + 5*(-0.1883)*(0.5270) - (-0.1883)^2 = -4.6273 * 10^{-3}$$

$$x_2 = x_1 - \frac{u_1 \frac{\partial v_1}{dy} - v_1 \frac{\partial u_1}{dy}}{\frac{\partial u_1}{dx}\frac{\partial v_1}{dy} - \frac{\partial u_1}{dy}\frac{\partial v_1}{dx}} = (-0.1883) - \frac{(7.5689*10^{-4})*(0.0585) - (-4.627*10^{-3})*(1)}{(-1.3766)*(0.0585) - (1)*(3.0116)} \cong 0.1868$$

$$y_2 = y_1 - \frac{v_1 \frac{\partial u_1}{dx} - u_1 \frac{\partial v_1}{dx}}{\frac{\partial u_1}{dx}\frac{\partial v_0}{dy} - \frac{\partial u_1}{dy}\frac{\partial v_0}{dx}} = 0.5270 - \frac{(-4.627*10^{-3})*(-1.3766) - (7.5689*10^{-4})*(3.0116)}{(-1.3766)*(0.0585) - 1*(3.0116)} \cong 0.5283$$