Nonlinear Systems of Equations, Gauss-Jordan [1-6]

References:

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the Gauss-Jordan method

$$2x_1 + 3x_2 - 2x_3 = 3$$
$$2x_1 - 2x_2 + 4x_3 = 18$$
$$-4x_1 + 2x_2 + x_3 = 3$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -2 & 3 \\ 2 & -2 & 4 & 18 \\ -4 & 2 & 1 & 3 \end{bmatrix}$$

Normalize the first row by dividing it by the pivot element, 2, to yield;

$$[1 \quad 1.5 \quad -1 \quad 1.5]$$

Subtract 2 times the first row from the second row. (Reduction of the x_1 term from the second row.)

$$[2 \quad -2 \quad 4 \quad 18] - (2)[1 \quad 1.5 \quad -1 \quad 1.5] = [0 \quad -5 \quad 6 \quad 15]$$

Subtract -4 times the first row from the third row. (Reduction of the x_1 term from the third row.)

$$[-4 \quad 2 \quad 1 \quad 3] - (-4)[1 \quad 1.5 \quad -1 \quad 1.5] = [0 \quad 8 \quad -3 \quad 9]$$

Normalize the second row by dividing it by -5.

$$[0 \quad 1 \quad -1.2 \quad -3]$$

Subtract 8 times the second row from the third row. (Reduction of the x_2 term from the third row.)

$$[0 \quad 8 \quad -3 \quad 9] - (8)[0 \quad 1 \quad -1.2 \quad -3] = [0 \quad 0 \quad 6.6 \quad 33]$$

Normalize the third row by dividing it by 6.6.

$$[0 \quad 0 \quad 1 \quad 5]$$

Subtract 1.5 times the second row from the first row. (Reduction of the x_2 term from the first row.)

$$[1 \quad 1.5 \quad -1 \quad 1.5] - (1.5)[0 \quad 1 \quad -1.2 \quad -3] = [1 \quad 0 \quad 0.8 \quad 6]$$

Subtract 0.8 times the third row from the first row. (Reduction of the x_3 term from the first row.)

$$[1 \quad 0 \quad 0.8 \quad 6] - (0.8)[0 \quad 0 \quad 1 \quad 5] = [1 \quad 0 \quad 0 \quad 2]$$

Subtract -1.2 times the third row from the second row. (Reduction of the x_3 term from the second row.)

$$[0 \quad 1 \quad -1.2 \quad -3] - (-1.2)[0 \quad 0 \quad 1 \quad 5] = [0 \quad 1 \quad 0 \quad 3]$$

Then the augmented matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow \boldsymbol{x_1 = 2} \quad \boldsymbol{x_2 = 3} \quad \boldsymbol{x_3 = 5}$$

Fixed Point Iteration For a Nonlinear System

$$u(x, y) = x^2 + xy - 10 = 0 \qquad (a)$$

$$v(x, y) = y + 3xy^2 - 57 = 0 \qquad (b)$$

$$x_{i+1} = \frac{10 - x_i^2}{y_i} \qquad (a')$$

$$y_{i+1} = 57 - 3x_i y_i^2 \qquad (b')$$

On the basis of the initial guesses, Eq. (a') can be used to determine a new value of x;

$$x = \frac{10 - (1.0)^2}{2.8} = 3.2143$$

This result and the initial value of $y = 2.8$ can be substituted into Eq. (b') to determine a new value of y:

$$y = 57 - (3)(3.2143)(2.8)^2 = -18.6003$$

$$x = \frac{10 - (3.2143)^2}{-18.6003} = 0.0178$$

diverging

$$y = 57 - (3)(0.0178)(-18.6003)^2 = 38.5198$$

Now, we will repeat the computation but with the original equations set up in a different format. For example, an alternative computation of Eq. (a) is

$$x = \sqrt{10 - xy}$$

and of Eq. (b) is

$$y = \sqrt{\frac{57 - y}{3x}}$$

Now the results are more satisfactory :

$$\left.\begin{aligned} x &= \sqrt{10-(1.0)(2.8)} = 2.6833 \\ y &= \sqrt{\frac{57-2.8}{(3)(2.6833)}} = 2.5948 \end{aligned}\right\} 1^{st}$$

$$\left.\begin{aligned} x &= \sqrt{10-(2.6833)(2.5948)} = 1.7428 \\ y &= \sqrt{\frac{57-2.5948}{(3)(1.7428)}} = 3.2258 \end{aligned}\right\} 2^{nd}$$

$$\left.\begin{aligned} x &= \sqrt{10-(1.7428)(3.2258)} = 2.0924 \\ y &= \sqrt{\frac{57-3.2258}{(3)(2.0924)}} = 2.9269 \end{aligned}\right\} 3^{rd}$$

$$\left.\begin{aligned} x &= \sqrt{10 - (2.0924)(2.9269)} = 1.9687 \\ y &= \sqrt{\frac{57 - 2.9269}{(3)(1.9687)}} = 3.0258 \end{aligned}\right\} 4^{th}$$

$$\left.\begin{aligned} x &= \sqrt{10 - (1.9687)(3.0258)} = 2.0107 \\ y &= \sqrt{\frac{57 - 3.0258}{(3)(2.0107)}} = 2.9913 \end{aligned}\right\} 5^{th}$$

Thus, the approach is converging on the true values of $x = 2$ and $y = 3$.

Newton-Raphson for a Nonlinear System

$$x_{i+1} = x_i - \frac{u_i \dfrac{\partial v_i}{\partial y} - v_i \dfrac{\partial u_i}{\partial y}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}} \qquad (a)$$

$$y_{i+1} = y_i - \frac{v_i \dfrac{\partial u_i}{\partial x} - u_i \dfrac{\partial v_i}{\partial x}}{\dfrac{\partial u_i}{\partial x}\dfrac{\partial v_i}{\partial y} - \dfrac{\partial u_i}{\partial y}\dfrac{\partial v_i}{\partial x}} \qquad (b)$$

$$u(x,y) = x^2 + xy - 10 = 0$$

$$v(x,y) = y + 3xy^2 - 57 = 0$$

$$\frac{\partial u_0}{\partial x} = 2x + y = (2)(1.0) + (2.8) = 4.8$$

$$\frac{\partial u_0}{\partial y} = x = 1.0$$

$$\frac{\partial v_0}{x} = 3y^2 = (3)(2.8)^2 = 23.52$$

$$\frac{\partial v_0}{\partial y} = 1 + 6xy = 1 + (6)(1.0)(2.8) = 17.8$$

Thus the determinant of the Jacobian for the first iteration is;

$$(4.8)(17.8) - (1.0)(23.52) = 61.92$$

The values of the functions can be evaluated at the initial guesses as

$$u_0 = (1.0)^2 + (1.0)(2.8) - 10 = -6.2$$

$$v_0 = (2.8) + (3)(1.0)(2.8)^2 - 57 = -30.68$$

These values can be substituted into Eq. (a) and (b) to give

$$x = 1.0 - \frac{(-6.2)(17.8) - (-30.68)(1.0)}{61.92} = 2.2868$$

$$y = 2.8 - \frac{(-30.68)(4.8) - (-6.2)(23.52)}{61.92} = 2.8233$$

2^{nd} iteration :

$$\frac{\partial u_1}{\partial x} = (2)(2.2868) + (2.8233) = 7.3969$$

$$\frac{\partial u_1}{\partial y} = 2.2868$$

$$\frac{\partial v_1}{\partial x} = (3)(2.8233)^2 = 23.9131$$

$$\frac{\partial v_1}{\partial y} = 1 + (6)(2.2868)(2.8233) = 39.7379$$

$$\text{determinant}_1 = (7.3969)(39.7379) - (2.2868)(23.9131) = 239.2528$$

$$u_1 = (2.2868)^2 + (2.2868)(2.8233) - 10 = 1.6858$$

$$v_1 = 2.8233 + (3)(2.2868)(2.8233)^2 - 57 = 0.5077$$

$$x = 2.2868 - \frac{(1.6858)(39.7379) - (0.5077)(2.2868)}{239.2528} = 2.0117$$

$$y = 2.8233 - \frac{(0.5077)(7.3969) - (1.6858)(23.9131)}{239.2528} = 2.9761$$

3^rd iteration :

$$\frac{\partial u_2}{\partial x} = (2)(2.0117) + (2.9761) = 6.9995$$

$$\frac{\partial u_2}{\partial y} = 2.0117$$

$$\frac{\partial v_2}{\partial x} = (3)(2.9761)^2 = 26.5715$$

$$\frac{\partial v_2}{\partial y} = 1 + (6)(2.0117)(2.9761) = 36.9221$$

$$D_2 = (6.9995)(36.9221) - (2.0117)(26.5715) = 204.9824$$

$$u_2 = (2.0117)^2 + (2.0117)(2.9761) - 10 = 0.0339$$

$$v_2 = 2.9761 + (3)(2.0117)(2.9761)^2 - 57 = -0.5699$$

$$x = 2.0117 - \frac{(0.0339)(36.9221) - (-0.5699)(2.0117)}{204.9824} = 2.0000$$

$$y = 2.9761 - \frac{(-0.5699)(6.9995) - (0.0339)(26.5715)}{204.9824} = 2.9999$$