# Least-Squares Regression, General linear least Squares, Nonlinear Regression [1-6]

**References:**


**1.** Chapra S.C. and Canale R.P. "Numerical Methods for Engineers", Sixth Edition,McGraw Hill, International Edition 2010.

**2.** Chapra S.C. and Canale R. P. "Yazılım ve programlama Uygulamalarıyla Mühendisler için Sayısal Yöntemler" 4.Basımdan Çevirenler: Hasan Heperkan ve Uğur Kesgin 2003.

**3.**Chapra S.C. "Applied Numerical Methods with MATLAB for engineers and Scientists" Third Edition, McGraw Hill, International Edition 2012.

**4.** Mathews J.H. and Fınk K.D. "Numerical Methods using MATLAB", Fourth Edition, Pearson P. Hall, International Edition 2004.

**5.** Fausett L.V. "Applied Numerical Analysis Using MATLAB, Second Edition, PearsonP. Hall, International Edition, 2008.

**6.** Gilat A. And Subramaniam V. "Numerical Methods, An introduction with Applications Using MATLAB", Second Edition,John Wiley and Sons. Inc. 2011.


$$y = \alpha e^{\beta x}$$

Take the natural logarithm of both sides.

$$\ln y = \ln \alpha + \beta x$$

This equation is in the form of $Y = a_1 X + a_0$ with $\ln \alpha$ corresponds to $a_0$ and $\beta$ corresponds to $a_1$.

$lny = Y$

$ln\alpha = a_0$

$\beta = a_1$

$x = X$

| X (x) | 1300 | 1500 | 1600 | 1700 | 1800 | 1900 |
|---|---|---|---|---|---|---|
| Y (ln y) | 4.0943 | 4.3820 | 4.6540 | 5.0752 | 5.7038 | 6.1738 |

The following quantities can be computed.

$$\Sigma X_i Y_i = 1300 * 4.0943 + 1500 * 4.382 + 1600 * 4.654 + 1700 * 5.0752 + 1800 * 5.7038 + 1900 * 6.1738 = 49966.89$$

$$\Sigma {X_i}^2 = 1300^2 + 1500^2 + 1600^2 + 1700^2 + 1800^2 + 1900^2 = 16240000$$

$$\Sigma X_i = 1300 + 1500 + 1600 + 1700 + 1800 + 1900 = 9800$$

$$\Sigma Y_i = 4.0943 + 4.382 + 4.654 + 5.0752 + 5.7038 + 6.1738 = 30.0831$$

$$\bar{X} = \frac{\Sigma X_i}{n} = \frac{9800}{6} = 1633.3333$$

$$\bar{Y} = \frac{\Sigma Y_i}{n} = \frac{30.0831}{6} = 5.0139$$

Using equations for $a_1$ and $a_0$;

$$a_1 = \frac{n\Sigma X_i Y_i - \Sigma X_i \Sigma Y_i}{n\Sigma {X_i}^2 - (\Sigma X_i)^2} = \frac{6 * 49966.89 - 9800 * 30.0831}{6 * 16240000 - 9800^2} = 0.0036$$

$$a_0 = \bar{Y} - a_1 \bar{X} = 5.0139 - 0.0036 * 1633.3333 = -0.8661$$

Thus, the linear least-squares regression yields $Y = 0.0036X - 0.8661$

Substituting $Y = \ln y$ gives $\ln y = 0.0036x - 0.8661$

Thus, the intercept, $\ln \alpha$, equals -0.8661 and therefore by taking the antilogarithm

$$\ln \alpha = -0.8661 \quad \rightarrow \quad \alpha = 0.4206$$

$$\beta = a_1 = 0.0036$$

Consequently, the exponential equation is $y = 0.4206e^{0.0036x}$

# linear least-squares regression

$$y = a_1 x + a_0$$

| $x_i$ | -6 | -4 | -1 | 0 | 3 | 5 | 8 |
|---|---|---|---|---|---|---|---|
| $y_i$ | 15 | 12 | 8 | 4 | 1 | -5 | -14 |

$$\Sigma\, x_i = (-6) + (-4) + (-1) + 0 + 3 + 5 + 8 = 5$$

$$\Sigma\, y_i = 15 + 12 + 8 + 4 + 1 + (-5) + (-14) = 21$$

$$\Sigma\, x_i{}^2 = (-6)^2 + (-4)^2 + (-1)^2 + 3^2 + 5^2 + 8^2 = 151$$

$$\Sigma\, x_i y_i = (-6) * 15 + (-4) * 12 + (-1) * 8 + 0 * 4 + 3 * 1 + 5 * (-5) + 8 * (-14) = -280$$

$$\bar{x} = \frac{\Sigma\, x_i}{n} = \frac{21}{7} = 3$$

$$\bar{y} = \frac{\Sigma\, y_i}{n} = \frac{5}{7} = 0.7143$$

Using equations for $a_1$ and $a_0$;

$$a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i{}^2 - (\Sigma x_i)^2} = \frac{7 * (-280) - 5 * 21}{7 * 151 - 5^2} = -2.0010$$

$$a_0 = \bar{y} - a_1 \bar{x} = 3 - (-2.0010) * 0.7143 = 4.4293$$

Therefore the least squares fit is $y = -2.0010x + 4.4293$

$$S_t = \Sigma(y_i - \bar{y})^2$$

| $y_i$ | $(y_i - \bar{y})^2$ |
|---|---|
| 15 | $(15 - 0.7143)^2 = 204.0812$ |
| 12 | $(12 - 0.7143)^2 = 127.3670$ |
| 8 | $(8 - 0.7143)^2 = 53.0814$ |
| 4 | $(4 - 0.7143)^2 = 10.7958$ |
| 1 | $(1 - 0.7143)^2 = 0.0816$ |
| -5 | $(-5 - 0.7143)^2 = 32.6532$ |
| -14 | $(-14 - 0.7143)^2 = 216.5106$ |

$S_t = \Sigma(y_i - \bar{y})^2 = 644.5708$

# linear least-squares regression

the coefficients α and β in the function $y = \left(\frac{1}{\alpha x^2+\beta}\right)^2$

| $x_i$ | -1.0 | -0.3 | 0.2 | 1.0 | 2.0 |
|---|---|---|---|---|---|
| $y_i$ | 2.8 | 4.4 | 4.6 | 2.7 | 1.2 |

The function is linearized by inverting to give

$$y = \left(\frac{1}{\alpha x^2 + \beta}\right)^2$$

$$\sqrt{y} = \frac{1}{\alpha x^2 + \beta} \quad \rightarrow \quad \frac{1}{\sqrt{y}} = \alpha x^2 + \beta \quad or \quad Y = a_1 X + a_0$$

where

$$\frac{1}{\sqrt{y}} = Y$$

$$\alpha = a_1$$

$$x^2 = X$$

$$\beta = a_0$$

| X ($x^2$) | 1 | 0.09 | 0.04 | 1 | 4 |
|---|---|---|---|---|---|
| Y ($\frac{1}{\sqrt{y}}$) | 0.5976 | 0.4767 | 0.4663 | 0.6086 | 0.9129 |

The following quantities can be computed.

$$\sum X_i Y_i = 1 * 0.5976 + 0.09 * 0.4767 + 0.04 * 0.4663 + 1 * 0.6086 + 4 * 0.9129 = 4.9194$$

$$\sum X_i^2 = 1^2 + 0.09^2 + 0.04^2 + 1^2 + 4^2 = 18.0097$$

$$\sum X_i = 1 + 0.09 + 0.04 + 1 + 4 = 6.13$$

$$\sum Y_i = 0.5976 + 0.4767 + 0.4663 + 0.6086 + 0.9129 = 3.0621$$

$$\bar{X} = \frac{\sum X_i}{n} = \frac{6.13}{5} = 1.226$$

$$\bar{Y} = \frac{\sum Y_i}{n} = \frac{3.0621}{5} = 0.6124$$

Using equations for $a_1$ and $a_0$;

$$a_1 = \frac{n\Sigma X_i Y_i - \Sigma X_i \Sigma Y_i}{n\Sigma X_i^2 - (\Sigma X_i)^2} = \frac{5 * 4.9194 - 6.13 * 3.0621}{5 * 18.0097 - 6.13^2} = 0.1110$$

$$a_0 = \bar{Y} - a_1\bar{X} = 0.6124 - 0.1110 * 1.226 = 0.4763$$

Thus, the linear least-squares regression yields

$$y = \left(\frac{1}{0.1110x^2 + 0.4763}\right)^2$$

# least-squares regression

$$y = a_2 x^2 + a_1 x + a_0$$

the coefficients $a_0$, $a_1$ and $a_2$

$$\Sigma x_i = 0 + 1 + 2 + 3 = 6$$

$$\Sigma y_i = 2.3 + 7 + 14 + 27 = 50.3$$

$$\Sigma x_i^2 = (0)^2 + (1)^2 + (2)^2 + (3)^2 = 14$$

$$\Sigma x_i^3 = (0)^3 + (1)^3 + (2)^3 + (3)^3 = 36$$

$$\Sigma x_i^4 = (0)^4 + (1)^4 + (2)^4 + (3)^4 = 36 = 98$$

$$\Sigma x_i y_i = 0 \times 2.3 + 1 \times 7 + 2 \times 14 + 3 \times 27 = 116$$

$$\Sigma x_i^2 y_i = (0)^2 \times 2.3 + (1)^2 \times 7.0 + (2)^2 \times 14 + (3)^2 \times 27 = 306$$

So the matrices become,

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 50.3 \\ 116 \\ 306 \end{Bmatrix}$$

Solving the system of linear equations through Gauss-Elimination gives,

$$[6 \quad 14 \quad 36 \quad 116] - \frac{6}{4}[4 \quad 6 \quad 14 \quad 50.3] = [0 \quad 5 \quad 15 \quad 40.55]$$

$$[14 \quad 36 \quad 98 \quad 306] - \frac{14}{4}[4 \quad 6 \quad 14 \quad 50.3] = [0 \quad 15 \quad 49 \quad 129.95]$$

$$[0 \quad 15 \quad 49 \quad 129.95] - \frac{15}{5}[0 \quad 5 \quad 15 \quad 40.55] = [0 \quad 0 \quad 4 \quad 8.3]$$

So, the matrix becomes

$$\begin{bmatrix} 4 & 6 & 14 \\ 0 & 5 & 15 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 50.3 \\ 40.55 \\ 8.3 \end{Bmatrix} \rightarrow \begin{matrix} a_2 = 2.075 \\ a_1 = 1.885 \\ a_0 = 2.485 \end{matrix}$$

$$2.075 + 1.885x + 2.485x^2$$

$$\bar{y} = \frac{\Sigma\, y_i}{n} = \frac{50.3}{4} = 12.575$$

| $x_i$ | $y_i$ | $S_t = (y_i - \bar{y})^2$ | $S_r = (y_i - a_0 - a_1 x_i - a_2 {x_i}^2)^2$ |
|---|---|---|---|
| 0 | 2.3 | 109.7256 | 0.0342 |
| 1 | 7.0 | 31.0806 | 0.3080 |
| 2 | 14 | 2.0306 | 0.3080 |
| 3 | 27 | 208.0806 | 0.0342 |
| $\sum$ | 50.3 | 350.9174 | 0.6844 |

$$r^2 = \frac{S_t - S_r}{S_t} = 0.9980$$