Lecture 10: Eigenvalues and Eigenvectors

Elif Tan

Ankara University

Eigenvalues and Eigenvectors

Definition (Eigenvalues and Eigenvectors)

Let $L : V \to V$ be a linear transformation and $dimV = n$. The scalar λ is called an eigenvalue of L if $\exists\ 0 \neq v \in V$ such that

$$L(v) = \lambda \odot v,$$

and the vector v is called an eigenvector of L associated with the eigenvalue λ.

In $\mathbb{R}^n$, the eigenvalue problem reduces to determine whether $\lambda \odot v$ can be parallel to v.

Eigenvalues and Eigenvectors

The eigenvalue problem for linear transformation can be stated as a matrix representation of this linear transformation.

Definition (Characteristic polynomial)

Let A be $n \times n$ matrix. The characteristic polynomial of A is defined by

$$P_A(\lambda) := \det(\lambda I_n - A).$$

The equation

$$P_A(\lambda) = \det(\lambda I_n - A) = 0$$

is called the characteristic equation of A.
The roots of the characteristic polynomial are eigenvalues of A.
Nonzero solutions of the homogenous linear system $(\lambda I_n - A)x = 0$ are called eigenvectors of A associated with the eigenvalue λ.

Eigenvalues and Eigenvectors

If we expand the determinant $P_A(\lambda)$ and collect terms in the same power of λ, we have

$$P_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_1\lambda + a_0.$$

Theorem (Cayley-Hamilton Theorem)

Every square matrix A satisfies its own characteristic equation, i.e.

$$P_A(A) = 0.$$

In the following, we give some applications of the Cayley-Hamilton Theorem.

1. $\det(A) = (-1)^n a_0$.
2. If $a_0 \neq 0$, then A^{-1} exists and $A^{-1} = \frac{-1}{a_0}\left(A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_2 A + a_1 I_n\right)$.
3. $A^n = -\left(a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A + a_0 I_n\right)$.

Example

Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_n - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda - 1 & -4 & 0 \\ 0 & \lambda - 2 & -5 \\ 0 & 0 & \lambda - 3 \end{vmatrix} = 0 \\ &\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda - 3) = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2$, and $\lambda_3 = 3$.

Eigenvalues and Eigenvectors

- To find the eigenvectors corresponding to the eigenvalue $\lambda_1 = 1$, we solve the equation $(\lambda I_n - A)\, x = 0$, *i.e.*

$$\begin{cases} (\lambda - 1)\, x_1 - 4x_2 = 0 \\ (\lambda - 2)\, x_2 - 5x_3 = 0 \\ \qquad\qquad (\lambda - 3)\, x_3 = 0 \end{cases}$$

where $\lambda = 1$. We find that $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}$, for $r \in \mathbb{R}$. That is, the eigenvectors corresponding to the eigenvalue $\lambda = 1$ are precisely the set of scalar multiples of the vector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvalues and Eigenvectors

- Similarly, the eigenvectors corresponding to the eigenvalue $\lambda = 2$ and $\lambda = 3$ are

$$v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix},$$

respectively.

Example

Find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution:

$$\begin{aligned} P_A(\lambda) &= \det(\lambda I_3 - A) = 0 \\ &\Rightarrow \begin{vmatrix} \lambda - 1 & 1 & 1 \\ 0 & \lambda - 3 & -2 \\ 0 & 1 & \lambda \end{vmatrix} = 0 \\ &\Rightarrow (\lambda - 1)(\lambda(\lambda - 3) + 2) = 0 \\ &\Rightarrow (\lambda - 1)^2 (\lambda - 2) = 0. \end{aligned}$$

The eigenvalues of A are $\lambda_{1,2} = 1$ (the multiplicity is 2) and $\lambda_3 = 2$.

Eigenvalues and Eigenvectors

- The eigenvectors corresponding to the eigenvalue $\lambda_{1,2} = 1$ are
  $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$
- The eigenvectors corresponding to the eigenvalue $\lambda_3 = 2$ is
  $v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.