Lecture 7: Normal Subgroups and Factor Groups

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Normal Subgroups

We know that if H is a subgroup of a group G, then G can be written as a disjoint union of distinct left (right) cosets of H in G. If left cosets are the same as right cosets, we call such subgroups as normal.

Definition (Normal Subgroup)

Let G be a group. A subgroup H of G is called a **normal subgroup** of G, written as $H \trianglelefteq G$, if $aH = Ha$ for all $a \in G$. That is,

$$H \trianglelefteq G \Leftrightarrow aH = Ha.$$

Note that $aH = Ha$ does not always mean that $ah = ha$ for all $h \in H, a \in G$.

Examples:

1. $\{e\} \trianglelefteq G$ (trivial normal subgroup)
2. $G \trianglelefteq G$ (proper normal subgroup)
3. $M(G) \trianglelefteq G$

Normal Subgroups

Theorem (Normal Subgroup Criterias)

Let G be a group and let $H \leq G$. Then the followings are equivalent conditions for H to be a normal subgroup of G:

1. $ghg^{-1} \in H$ *for all* $g \in G$ *and all* $h \in H$.
2. $gHg^{-1} = H$ *for all* $g \in G$.
3. $gH = Hg$ *for all* $g \in G$.

Theorem

Let G be an **abelian** *group and let $H \leq G$. Then $H \trianglelefteq G$.*

Example: Every subgroup of $\mathbb{Z}$ is of form $n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$. Since $\mathbb{Z}$ is an abelian group, every subroup of $\mathbb{Z}$ is normal.

Normal Subgroups

Theorem

Let G be a group and let $H, K \trianglelefteq G$. Then

1. $H \cap K \trianglelefteq G$
2. $HK \trianglelefteq G$
3. $HK = \langle H \cup K \rangle$.

Remark: Let G be a group.

- If $H \leq G$ and $K \leq G \Rightarrow HK$ need not be a subgroup of G.
  Let $H \leq G$ and $K \leq G$. Then

  $$HK \leq G \Leftrightarrow HK = \langle H \cup K \rangle \Leftrightarrow HK = KH$$

  Let $H \leq G$ and $K \leq G$. If G is abelian $\Rightarrow HK \leq G$.
- If $H \trianglelefteq G$ and $K \leq G \Rightarrow HK \leq G$
- If $H \leq G$ and $K \trianglelefteq G \Rightarrow HK \leq G$
- If $H \trianglelefteq G$ and $K \trianglelefteq G \Rightarrow HK \trianglelefteq G$

Normal Subgroups

- Let G be a group and let $H, K \trianglelefteq G$ and $H \cap K = \{e\}$. Then $hk = kh$ for all $h \in H$ and $k \in K$.
- Let G be a group and let $H \leq G$. If $[G : H] = 2$, then $H \trianglelefteq G$.
- If H is the only subgroup of order n in a group G, then $H \trianglelefteq G$.

Normal Subgroups

- Let G be a cyclic group. Then G is abelian. Thus every subgroup of G is normal.
  Now we determine all normal subgroups of $\mathbb{Z}_n$. Let N be a normal subgroup of $\mathbb{Z}_n$. We know that each subgroup of $\mathbb{Z}_n$ is cyclic, since $\mathbb{Z}_n = \langle \overline{1} \rangle$. Hence, $N = \langle \overline{a} \rangle$ is a normal subgroup of $\mathbb{Z}_n \Leftrightarrow a \mid n$.
  Example: Find all normal subgroups of $\mathbb{Z}_{12}$. $N = \langle \overline{a} \rangle$ is a normal subgroup $\mathbb{Z}_{12} \Leftrightarrow a \mid 12$. Thus the normal subgroups of $\mathbb{Z}_{12}$ are

$$\begin{aligned}
\langle \overline{1} \rangle &= \mathbb{Z}_{12} \\
\langle \overline{2} \rangle &= \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}, \overline{10}\} \\
\langle \overline{3} \rangle &= \{\overline{0}, \overline{3}, \overline{6}, \overline{9}\} \\
\langle \overline{4} \rangle &= \{\overline{0}, \overline{4}, \overline{8}\} \\
\langle \overline{6} \rangle &= \{\overline{0}, \overline{6}\} \\
\langle \overline{12} \rangle &= \langle \overline{0} \rangle = \{\overline{0}\} .
\end{aligned}$$

Factor Groups

Let G be a group and $H \trianglelefteq G$. For $a, b \in H$, the relation $\sim$ defined by "$a \sim b \Leftrightarrow a^{-1}b \in H$" is an equivalence relation on G. Let denote the set of all equivalence classes as

$$G/H := \{aH \mid a \in G\}.$$

Theorem

Let G be a group and $H \trianglerighteq G$. Define a binary operation $\odot$ on G/H by

$$(aH) \odot (bH) := (ab)\, H$$

for $aH, bH \in G/H$. Then $(G/H, \odot)$ is a group.

Definition

The group $(G/H, \odot)$ is called the **factor(quotient) group** of G by H.

Factor Groups

Remarks:

- If G is a commutative group, then G/H is also commutative. For all $aH, bH \in G/H$,

$$(aH)(bH) = (ab)H = (ba)H = (bH)(aH)$$

- If G is a cyclic group and $H \leq G$. Then G/H is cyclic.
- If G is a group and $G/M(G)$ is cyclic, then G is commutative. Since Q_8 is not commutative, then $Q_8/M(Q_8)$ is not cyclic. Also $M := M(Q_8) = \{\pm 1\}$ and

$$Q_8/M(Q_8) = \{M, iM, jM, kM\}.$$

  Since there is no element of order 4 in $Q_8/M(Q_8)$, $Q_8/M(Q_8)$ is not cyclic.
  Note that every subgroup of Q_8 is normal.

Factor Groups

- Let G be a group and $N \trianglelefteq G$. Then every subgroup of G/N is of the form K/N where $N \subseteq K \leq G$. That is,

$$
\begin{aligned}
K/N &\leq G/N \; ; \; N \subseteq K \leq G \\
K/N &\trianglelefteq G/N \; ; \; N \subseteq K \trianglelefteq G.
\end{aligned}
$$

  For details, see the Correspondence Theorem.
- G is **simple group** if its only normal subgroups are $\{e\}$ and G.