# Lecture 8: Group Homomorphisms and Isomorphism Theorems

Prof. Dr. Ali Bülent EKİN
Doç. Dr. Elif TAN

Ankara University

# Group Homomorphisms

## Definition

Let $(G, *)$ and $(G', *')$ be two groups. A function $f : G \to G'$ is called a **homomorphism** from $G$ into $G'$ if for all $a, b \in G$,

$$f(a * b) = f(a) *' f(b).$$

- If $f$ is one-to-one, then $f$ is called a **monomorphism**.
- If $f$ is onto, then $f$ is called an **epimorphism**.($G'$ is called the **homomorphic image** of $G$).
- If $f$ is one-to-one and onto, then $f$ is called an **isomorphism**. The groups $G$ and $G'$are called isomorphic and denoted by $G \simeq G'$.
- An isomorphism from the ring $G$ onto $G$, is called an **automorphism**.
- Let $f : G \to G'$ be a group homomorphism. Then $Kerf := \{g \in G \mid f(g) = e'\} = f^{-1}(\{e'\})$ is called the **kernel** of $f$, $f(G) := \{f(g) \mid g \in G\}$ is called the **image** of $f$.

# Properties of Group Homomorphisms

## Theorem

*Let $f : G \longrightarrow G'$ be a group homomorphism. Then we have*

1. $f(e) = e'$.
2. $f(a^{-1}) = f(a)^{-1}$ *for all* $a \in G$.
3. *If* $H \leq G$, *then* $f(H) \leq G'$.
4. *If* $H' \leq G'$, *then* $f^{-1}(H') \leq R$.
5. *If* $N \trianglelefteq G$ *and* $f$ *is* **onto**, *then* $f(N) \trianglelefteq G'$.
6. *If* $N' \trianglelefteq G'$, *then* $f^{-1}(N') \trianglelefteq G$.
7. *Kerf* $\trianglelefteq G$.
8. *Ker* $f = \{e\} \Leftrightarrow f$ *is one-to-one.*
9. *If* $G$ *is commutative, then* $f(G)$ *is commutative.*
10. *If* $G$ *is commutative and* $f$ *is* **onto**, *then* $G'$ *is commutative.*
11. *If* $a \in G$ *such that* $\circ(a) = n$, *then* $\circ(f(a)) \mid n$.

**Examples:**

**1.** Let $f: G \to G'$, $f(a) = e'$ for all $a \in G$.Then $f$ is a (trivial)homomorphism such that $Kerf = G$.

**2.** Let $f$ be an identity map. Then $f$ is an isomorphism such that $Kerf = \{e\}$.

**3.** Let $f: \mathbb{Z} \to 2\mathbb{Z}$, $f(a) = 2a$ for all $a \in \mathbb{Z}$. Then $(\mathbb{Z}, +) \simeq (2\mathbb{Z}, +)$.

**4.** Let $f: \mathbb{Z}_6 \to \mathbb{Z}_{10}$, $f(\overline{a}) = 5\overline{a}$ for all $\overline{a} \in \mathbb{Z}_6$. Then $f$ is a homomorphism with $Kerf = \{\overline{0}, \overline{2}, \overline{4}\}$.

**5.** Let $f: \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $f(a) = (a, 0)$ for all $a \in \mathbb{Z}$. Then $f$ is a homomorphism with $\mathrm{Ker}f = \{0\}$.

**6.** No homomorphism $\mathbb{Z}_4 \times \mathbb{Z}_4 \to \mathbb{Z}_8 \times \mathbb{Z}_2$ since no element of order 8 in $\mathbb{Z}_4 \times \mathbb{Z}_4$.

**Theorem (Natural Homomorphism)**

*Let $G$ be a ring and $N \trianglelefteq G$. Then the function $\gamma : G \to G/N$ defined by $\gamma(g) = gN$ is an epimorphism with $Ker\gamma = N$. The homomorphism $\gamma$ is called the* **natural homomorphism** *of $G$ onto $G/N$.*

Note that

$$\begin{aligned} Kerf &= \{a \in G \mid f(a) = e_{G/N}\} \\ &= \{a \in G \mid aN = N\} \\ &= \{a \in G \mid a \in N\} = N. \end{aligned}$$

**Example:** The function $\gamma : \mathbb{Z} \to \mathbb{Z}/\langle n \rangle$ defined by $\gamma(a) = a + \langle n \rangle$ for all $a \in \mathbb{Z}_n$, is the natural homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}/\langle n \rangle$.

# Properties of Group Isomorphisms

## Theorem

*Let $f : G \longrightarrow G'$ be a group isomorphism. Then we have*

1. $f^{-1} : G' \longrightarrow G$ *is a group isomorphism.*
2. $\circ(a) = \circ(f(a))$ *for all* $a \in G$.
3. $G$ *is a commutative group* $\Leftrightarrow$ $G'$ *is commutative group.*
4. $G$ *is a torsion group* $\Leftrightarrow$ $G'$ *is a torsion group.*
5. $G$ *is a cyclic group* $\Leftrightarrow$ $G'$ *is a cyclic group.*

**Examples:**

**1.** Let $f : \mathbb{R} \to \mathbb{R}^+$, $f(a) = e^a$ for all $a \in \mathbb{R}$. Then $(\mathbb{R}, +) \simeq (\mathbb{R}^+, .)$.

**2.** $(\mathbb{Z}, +) \not\simeq (\mathbb{Q}, +)$ since $\mathbb{Z}$ is cyclic, but $\mathbb{Q}$ is not cyclic.

**3.** $(\mathbb{R}^*, .) \not\simeq (\mathbb{C}^*, .)$ since $\mathbb{R}^*$ does not have any element of order 4, but $\circ(i) = 4$ in $\mathbb{C}^*$.

**4.** $(\mathbb{Q}, +) \not\simeq (\mathbb{Q}^*, .)$ since every nonzero element in $\mathbb{Q}$ has infinite order, but $\circ(-1) = 2$ in $\mathbb{Q}^*$.

**5.** $(\mathbb{Q}, +) \not\simeq (\mathbb{Q}/\mathbb{Z}, +)$ since every nonzero element in $\mathbb{Q}$ has infinite order, but $\mathbb{Q}/\mathbb{Z}$ has not.

**6.** $(U_8, .) \simeq (U_{12}, .)$ where $U_n = \{\overline{a} \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$.

**7.** $(U_8, .) \not\simeq (U_{10}, .)$ since $U_8 = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}$ is not cyclic, but $U_{10} = \{\overline{1}, \overline{3}, \overline{7}, \overline{9}\}$ is cyclic.

The following theorem also known as **the fundamental homomorphism theorem**.

**Theorem (First Isomorphism Theorem)**

*Let $f : G \to G'$ be a homomorphism with $Kerf = N$. Then the function $\mu : G/N \to f(G)$ defined by $\mu(gN) = f(g)$ is an isomorphism; i.e. $G/Kerf \simeq f(G)$. Moreover, if $\gamma : G \to G/N$ is the natural homomorphism, then $f(g) = \mu\gamma(g)$, for each $g \in G$.*

**Example:** Let $f : \mathbb{Z} \to \mathbb{Z}_n$, $f(a) = \overline{a}$ for all $a \in \mathbb{Z}$. Then $f$ is an epimorphism such that

$$\begin{aligned} Kerf &= \{a \in \mathbb{Z} \mid f(a) = \overline{0}\} = \{a \in \mathbb{Z} \mid \overline{a} = \overline{0}\} \\ &= \{a \in \mathbb{Z} \mid a \equiv 0 \pmod{n}\} = \{nk \mid k \in \mathbb{Z}\} = n\mathbb{Z} = \langle n \rangle . \end{aligned}$$

From 1. isomorphism theorem, $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$. Moreover, let $\gamma : \mathbb{Z} \overset{natural\ hom.}{\longrightarrow} \mathbb{Z}/\langle n \rangle$, $a \longrightarrow a + \langle n \rangle$, then we have $f = \mu \circ \gamma$ st. $\mu : \mathbb{Z}/\langle n \rangle \overset{iso.}{\longrightarrow} \mathbb{Z}_n$, $a + \langle n \rangle \longrightarrow \overline{a}$.

**Theorem (Second Isomorphism Theorem)**

*Let $H \leq G$ and $N \trianglelefteq G$. Then*

$$H/\left(H \cap N\right) \simeq \left(HN\right)/N.$$

**Proof of synopsis:** Define $f : H \to \left(HN\right)/J$ by $f\left(h\right) = hN$. It is obvious that $f$ is an epimorphism with

$$Kerf = \left\{h \in H \mid f\left(h\right) = e_{(HN)/N}\right\} = \left\{h \in H \mid hN = N\right\} = H \cap N.$$

From 1. isomorphism theorem, $H/\left(H \cap N\right) \simeq \left(HN\right)/N$.

**Example:** Let $H = 6\mathbb{Z} \leq \mathbb{Z}$ and $N = 10\mathbb{Z} \trianglelefteq \mathbb{Z}$.

$$\begin{aligned} H + N &= 2\mathbb{Z} \Rightarrow \left(H+N\right)/N = 2\mathbb{Z}/10\mathbb{Z} = \left\{2k + 10\mathbb{Z} \mid k \in \mathbb{Z}\right\} \\ &= \left\{0 + 10\mathbb{Z}, 2 + 10\mathbb{Z}, 4 + 10\mathbb{Z}, 6 + 10\mathbb{Z}, 8 + 10\mathbb{Z}\right\} \\ H \cap N &= 30\mathbb{Z} \Rightarrow H/\left(H \cap N\right) = 6\mathbb{Z}/30\mathbb{Z} = \left\{6k + 30\mathbb{Z} \mid k \in \mathbb{Z}\right\} \\ &= \left\{0 + 30\mathbb{Z}, 6 + 30\mathbb{Z}, 12 + 30\mathbb{Z}, 18 + 30\mathbb{Z}, 24 + 30\mathbb{Z}\right\}. \end{aligned}$$

**Theorem (Third Isomorphism Theorem)**

*Let $H \trianglelefteq G$ and $K \trianglelefteq G$. such that $K \subseteq H$. Then*

$$(G/K)/(H/K) \simeq G/H.$$

**Proof of synopsis:** Define $f : G/K \rightarrow G/H$ by $f(gK) = gH$. It is obvious that $f$ is an epimorphism with

$$\begin{aligned} Kerf &= \{gK \in G/K \mid f(gK) = e_{G/H}\} \\ &= \{gK \in G/K \mid gH = H\} \\ &= \{gK \in G/K \mid g \in H\} = H/K. \end{aligned}$$

From 1. isomorphism theorem, $(G/K)/(H/K) \simeq G/H$.

**Example:** $(\mathbb{Z}/12\mathbb{Z})/(3\mathbb{Z}/12\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z}$.

# Correspondence Theorem

## Theorem (Correspondence Theorem)

*Let $f : G \longrightarrow G'$ be an epimorphism. Then there is a one-to-one correspondence between the normal subgroups of $G$ containing $Kerf$ and the normal subgroups of $G'$. That is, if $N \trianglelefteq G$ such that $Kerf \subseteq N$, then the corresponding normal subgroup is $f(N) \trianglelefteq G'$ and if $N' \trianglelefteq G'$, then the corresponding normal subgroup is $f^{-1}(N') = \{x \in G \mid f(x) \in N'\} \trianglelefteq G$.*

By Correspondence Theorem,

- There is a one-to-one correspondence between the normal subgroups of $G$ containing $N$ and the normal subgroups of the quotient group $G/N$.
- If $N \trianglelefteq G$. Then every subgroup of $G/N$ is of the form $K/N$ where $N \subseteq K \leq G$. That is,

$$K/N \leq G/N \Leftrightarrow N \subseteq K \leq G$$
$$K/N \trianglelefteq G/N \Leftrightarrow N \subseteq K \trianglelefteq G.$$

# Correspondence Theorem

**Example:** Consider the epimorphism $f: \begin{array}{c} \mathbb{Z} \to \mathbb{Z}_{12} \\ a \to \bar{a} \end{array}$ with $Kerf = 12\mathbb{Z}$.

All normal subroups of $\mathbb{Z}$ containing $12\mathbb{Z}$ are $\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle, \langle 6 \rangle, \langle 12 \rangle$.

All normal subroups of $\mathbb{Z}_{12}$ are $\langle \bar{0} \rangle, \langle \bar{1} \rangle, \langle \bar{2} \rangle, \langle \bar{3} \rangle, \langle \bar{4} \rangle, \langle \bar{6} \rangle$.

Thus we have

$$\begin{array}{ccl} \langle 1 \rangle & \leftrightarrow & \langle \bar{1} \rangle, \quad \langle 2 \rangle \leftrightarrow \langle \bar{2} \rangle, \quad \langle 3 \rangle \leftrightarrow \langle \bar{3} \rangle, \\ \langle 4 \rangle & \leftrightarrow & \langle \bar{4} \rangle, \quad \langle 6 \rangle \leftrightarrow \langle \bar{6} \rangle, \quad \langle 12 \rangle \leftrightarrow \langle \bar{0} \rangle. \end{array}$$