Lecture 10: Dihedral Groups

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One can construct a group from any geometric shape by the symmetries of it. The **symmetries** are the transformations, such as rotation and reflection, that maps the shape onto itself, so it looks the same before and after these transformations.
The group of all transformations under which the object is invariant is called the **group of symmetries**. When the shape is regular polygon the group is called the **dihedral group**.

Definition

The n-th **dihedral group** D_n is the group of symmetries of the regular n-gon. $(n \geq 3)$

Dihedral Groups

When we look at the symmetries for a regular n-gon;

- **Identity:** The transformation which makes no changes is e.
- **Rotation:** Let the size of the opposite clockwise rotation "r" is $\theta = \frac{360^\circ}{n}$, then there exists n-rotations $r, r^2, r^3, \ldots, r^n = e$.
- **Reflection:** Let consider the reflection "f" w.r.t. the vertical axis, then there exists 2-reflections $f, f^2 = e$.
- The remaining symmetries can be obtained by combining r and f as $\left\{rf, r^2 f, \ldots, r^{n-1} f\right\}$.
- Thus we have $2n$ different symmetries for a regular n-gon:

  $$\left\{e, r, r^2, \ldots, r^{n-1}, f, rf, r^2 f, \ldots, r^{n-1} f\right\}$$

  which can also be expressed as

  $$D_n = \left\langle r, f \mid r^n = f^2 = e, rf = fr^{n-1}\right\rangle.$$

- $|D_n| = 2n$
- D_n is non commutative, since $rf \neq fr$.

Example: Find the symmetry group for equilateral triangle.

$$D_3 = \{e, r, r^2, f, rf, r^2 f\}$$

$$\begin{aligned} e &\leftrightarrow (1) \\ r &\leftrightarrow (123) \\ r^2 &\leftrightarrow (132) \\ f &\leftrightarrow (23) \\ rf &\leftrightarrow (13) \\ r^2 f &\leftrightarrow (12) \end{aligned}$$

Thus $D_3 \simeq S_3$.

Example: Find the symmetry group for square.

$$D_4 = \{e, r, r^2, r^3, f, rf, r^2f, r^3f\}$$

$$\begin{aligned}
e &\leftrightarrow (1) \\
r &\leftrightarrow (1234) \\
r^2 &\leftrightarrow (13)(24) \\
r^3 &\leftrightarrow (1432) \\
f &\leftrightarrow (12)(34) \\
rf &\leftrightarrow (24) \\
r^2f &\leftrightarrow (14)(23) \\
r^3f &\leftrightarrow (13)
\end{aligned}$$

Example: Find the symmetry group for rectangle.

$$K = \{e, r, f, rf\}$$

$$\begin{aligned} e &\leftrightarrow (1) \\ r &\leftrightarrow (13)\,(24) \\ f &\leftrightarrow (12)\,(34) \\ rf &\leftrightarrow (14)\,(23) \end{aligned}$$

Thus $K \simeq V_4$. Note that this group is abelian.

Cayley's Theorem

Theorem

Every group is isomorphic to some group of permutations. That is any group can be seen as a permutation group.

Proof of synopsis:

- Define a function $\lambda_x : \underset{g \longrightarrow xg}{G \longrightarrow G}$ for all $g \in G$. Since λ_x is 1-1 and onto, it is a permutation.
- Define a function $\phi : \underset{x \longrightarrow \lambda_x}{G \longrightarrow S_G}$ for all $x \in G$. Since ϕ is 1-1 homomorphism, then $G \simeq \phi(G) \leq S_G$.

From Cayley's Theorem, let $V_4 = \{e, a, b, ab\}$.
For $x \in V_4$, $\lambda_x : \underset{g \longrightarrow xg}{V_4 \to V_4}$ and $\phi : \underset{x \longrightarrow \lambda_x}{V_4 \longrightarrow S_4}$, we have

$$\begin{aligned}
\lambda_e &= \begin{pmatrix} e & a & b & ab \\ e & a & b & ab \end{pmatrix} \leftrightarrow (1) \\
\lambda_a &= \begin{pmatrix} e & a & b & ab \\ a & e & ab & b \end{pmatrix} \leftrightarrow (12)\,(34) \\
\lambda_b &= \begin{pmatrix} e & a & b & ab \\ b & ab & e & a \end{pmatrix} \leftrightarrow (13)\,(24) \\
\lambda_{ab} &= \begin{pmatrix} e & a & b & ab \\ ab & b & a & e \end{pmatrix} \leftrightarrow (14)\,(23)
\end{aligned}$$

Thus $V_4 \simeq \phi(V_4) \leq S_4$.

Conjugacy Classes

Conjugacy classes help us to obtain a decomposition of the order of a finite group, called as class equation, which is very useful to determine the structure of a finite group.

Definition

Let G be a group and $a \in G$. An element $b \in G$ is said to be a **conjugate** of a in G denoted $a \approx b$, if there exists $x \in G$ such that $b = xax^{-1}$. That is, for $a, b \in G$

$$a \approx b \Leftrightarrow b = xax^{-1}, \exists x \in G.$$

- The conjugacy relation $\approx$ is an equivalence relation.
- The equivalence class of a is called a **conjugacy class** of a in G, denoted by $C(a)$. So

$$C(a) := \bar{a} = \{b \in G \mid b \approx a\} = \left\{xax^{-1} \mid x \in G\right\}, |C(a)| =: c_a.$$

- If G is abelian, then $C(a) = \left\{xax^{-1} \mid x \in G\right\} = \left\{axx^{-1} \mid x \in G\right\} = \{a\}$.

Conjugacy Classes

- $G = C(a_1) \cup C(a_2) \cup \cdots \cup C(a_k)$
  $\overset{G \text{ is finite}}{\Longrightarrow} |G| = c_{a_1} + c_{a_2} + \cdots + c_{a_k}$.
- $c_a = [G : M_G(a)]$ where

$$M_G(a) = \{x \in G \mid xa = ax\} = \{x \in G \mid xax^{-1} = a\}$$

  is the centralizer of a in G.
- **Class Equation:** Let G be a finite group. Then

$$|G| = |M(G)| + \sum_{a \notin M(G)} [G : M_G(a)]$$

- $a \in M(G) \Leftrightarrow M_G(a) = G \Leftrightarrow [G : M_G(a)] = 1$
- $|G| = p^n \Rightarrow |M(G)| > 1$
- $|G| = p^2 \Rightarrow G$ is abelian.

Conjugacy Classes

- Let $\pi = (i_1 i_2 \ldots i_r) \in S_n$ be a cycle. Then for all $\sigma \in S_n$,

$$\sigma\pi\sigma^{-1} = (\sigma(i_1)\,\sigma(i_2)\ldots\sigma(i_r)).$$

  This shows how to compute the conjugate.
- Two cycles in S_n are conjugate $\Leftrightarrow$ they have the same length.
- The number of conjugacy classes in S_n is $p(n)$, number of partitions of n.

Example: Find the conjugacy classes of S_3.
Since $p(3) = 3, 2+1, 1+1+1$ there exits 3 conjugacy classes.

cycle type	
$(i_1 i_2 i_3)$	$\overline{(123)} = \{(123), (132)\}$
$(i_1 i_2)(i_3)$	$\overline{(12)} = \{(23), (13), (12)\}$
$(i_1)(i_2)(i_3)$	$\overline{(1)} = \{(1)\} = M(S_3)$

Thus

$$|S_3| = 1 + 2 + 3.$$

Conjugacy Classes

- Note that

$$\begin{aligned}\overline{(123)} &= \left\{\sigma(123)\sigma^{-1} \mid \sigma \in S_3\right\} \\ &= \left\{\sigma(1)\,\sigma(2)\,\sigma(3) \mid \sigma \in S_3\right\} \\ &= \left\{(123),(132)\right\}.\end{aligned}$$

- Let $H \leq G$. Then $H \trianglelefteq G \Leftrightarrow H = \bigcup_{h \in H} \overline{h}$ where $\overline{h}$ is conjugacy class of h.

  From Lagrange's theorem $H \leq S_3 \Rightarrow |H| \mid |S_3|$, then $|H|$ could be $1, 2, 3, 6$.

  On the other hand, a normal subgroup must be a union of conjugacy classes of its elements, that is, $H = \bigcup_{h \in H} \overline{h}$ and H must contain the identity, $|H| = 1 + 2 = 3$. So $A_3 = \langle (123) \rangle$ is the only nontrivial normal subgroup of S_3. Observe that the class equation helps us to find such this normal subgroup.