Lecture 13: Sylow Theorems

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Sylow Theorems

We know that the fundamental theorem of finite abelian groups gives us information about all finite abelian groups. Now we give the Sylow's theorem which gives us information about finite nonabelian groups.

From the Lagrange's theorem,

$$G \text{ is a finite group and } H \leq G \Rightarrow |H| \mid |G| .$$

The converse of the Lagrange's theorem is not always true. Recall that $|A_4|$ has no subgroup of order 6. We also know that

$$\text{If } G \text{ is cyclic group and } H \leq G \Leftrightarrow |H| \mid |G| .$$

Now we show that the Sylow theorems give a weak converse of the Lagrange's theorem.

p-Groups

Definition

A group G is a p-group if every element in G has order a power of the prime p.
A subgroup of G is a p-subgroup of G if the subgroup is itself a p-group.

Theorem (Cauchy's Theorem)

Let G be a finite group.

$$p \mid |G| \Rightarrow G \text{ has an element of order } p, \text{ whence a subgroup of } p$$

- Let G be a finite group.

$$G \text{ is a } p\text{-group} \Leftrightarrow |G| \text{ is a power of a prime}$$

- Let G be a finite abelian group.

$$\text{If } m \mid |G| \Rightarrow \exists H \leq G \text{ such that } |H| = m.$$

p-Groups

Examples:

1. $\mathbb{Z}_6$ is not a *p*-group, since $|\mathbb{Z}_6| = 6 = 2.3$ is not a power of a prime p. But the subgroup $\langle \overline{3} \rangle = \{\overline{0}, \overline{3}\} \leq \mathbb{Z}_6$ is a 3-subgroup of $\mathbb{Z}_6$.

2. $V_4 = \left\{ \underbrace{e}_{2^0}, \underbrace{a}_{2^1}, \underbrace{b}_{2^1}, \underbrace{ab}_{2^1} \right\}$ is a 2-group.

3. $Q_8 = \left\{ \underbrace{1}_{2^0}, \underbrace{-1}_{2^1}, \underbrace{\pm i}_{2^2}, \underbrace{\pm j}_{2^2}, \underbrace{\pm k}_{2^2} \right\}$ is a 2-group.

Theorem (First Sylow Theorem)

Let G be a finite group of order $p^r m$. $(\gcd(p, m) = 1)$ Then G has a subgroup of order p^k for all k such that $0 \leq k \leq r$.

Definition

Let G be a finite group. A subgroup P of G is called a **Sylow p-subgroup** of G, if P is a maximal p-subgroup.

Example: Consider the symmetric group $S_3 = \{(1), (123), (132), (12), (13), (23)\}$. the Sylow 2-subgroups are

$$
\begin{aligned}
H_1 &= \{(1), (12)\} \leq S_3 \\
H_2 &= \{(1), (13)\} \leq S_3 \\
H_3 &= \{(1), (23)\} \leq S_3.
\end{aligned}
$$

- For each prime p, a finite group G has a Sylow p-subgroup
- Let $|G| = p^r m$.
  1. $H \leq G$ is a Sylow p-subgroup $\Leftrightarrow |H| = p^r$
  2. $P \leq G$ is a Sylow p-subgroup $\Rightarrow aPa^{-1} \leq G$ is a Sylow p-subgroup
  3. $P \leq G$ is the only Sylow p-subgroup $\Rightarrow P \trianglelefteq G$

Theorem (Second Sylow Theorem)

Any two Sylow p-subgroups of G are conjugate, that is, if P is a Sylow p-subgroup then every conjugate aPa^{-1} of P is also a Sylow p-subgroup.

Theorem (Third Sylow Theorem)

If G is a finite group and $p \mid |G|$, then the number of Sylow p-subgroups of G, n_p, is

1. $n_p \equiv 1 \pmod{p}$
2. $n_p \mid |G|$.

Sylow Theorems

Example: $|G| = 15 = 3.5$

- There exists Sylow 3-subgroup P_3
  $n_3 \equiv 1 \pmod{3}$ and $n_3 \mid 15 \Rightarrow n_3 = 1$
- There exists Sylow 5-subgroup P_5
  $n_5 \equiv 1 \pmod{5}$ and $n_5 \mid 15 \Rightarrow n_5 = 1$

Since G has exactly one subgroup of order 5, then $P_5 \trianglelefteq G$. Thus G is not simple.

Applications of Sylow Theorems

- $|G| = p \Rightarrow G$ is cyclic
- $|G| = p^2 \Rightarrow G$ is abelian
- $|G| = pq \Rightarrow G$ is not simple, $G \simeq \mathbb{Z}_p \times \mathbb{Z}_q$
- $|G| = 2p \Rightarrow G$ is either isomorphic to a cyclic group of order $2p$ or to D_p

Applications of Sylow Theorems

Determine all groups of order $1 \leq n \leq 10$.

n	abelian	nonabelian
1	$\{e\}$	-
2	$\mathbb{Z}_2$	-
3	$\mathbb{Z}_3$	-
4	$\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \simeq V_4$	-
5	$\mathbb{Z}_5$	-
6	$\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$	D_3
7	$\mathbb{Z}_7$	-
8	$\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	D_4, Q_8
9	$\mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3$	-
10	$\mathbb{Z}_{10} \simeq \mathbb{Z}_2 \times \mathbb{Z}_5$	D_5

Applications of Sylow Theorems

- Give information about the structure of a finite group.
- Give a weak converse of the Lagrange's theorem.
  By the help of the Cauchy's theorem,

  If G is a finite abelian group and $|H| \mid |G| \Rightarrow H \leq G$.

  By the help of the Sylow theorems,

  If G is a finite group and $p^k \mid |G| \Rightarrow \exists H \leq G$ such that $|H| = p^k$.

- Helps to find some finite groups which are not simple.
  A_5 is the smallest noncommutative simple group.