Lecture 5: Ring Homomorphisms and Isomorphisms

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Ring Homomorphisms

Definition

Let $(R, +, .)$ and $(R', +', .')$ be two rings. A function $f : R \longrightarrow R'$ is called a **homomorphism** from R into R' if
(i) $f(a+b) = f(a) +' f(b)$
(ii) $f(a.b) = f(a) .' f(b)$ for all $a, b \in R$.

- If f is one-to-one, then f is called a **monomorphism**.
- If f is onto, then f is called an **epimorphism**.(R' is called the **homomorphic image** of R).
- If f is one-to-one and onto, then f is called an **isomorphism**. The rings R and $R'$are called isomorphic and denoted by $R \simeq R'$.
- An isomorphism from the ring R onto R, is called an **automorphism**.
- Let $f : R \longrightarrow R'$ be a ring homomorphism. Then $\mathrm{Ker}f := \{r \in R \mid f(r) = 0_{R'}\} = f^{-1}(0_{R'})$ is called the **kernel** of f, $f(R) := \{f(r) \mid r \in R\}$ is called the **image** of f.

Properties of Ring Homomorphisms

Theorem

Let $f : R \longrightarrow R'$ be a ring homomorphism. Then we have

1. $f(0_R) = 0_{R'}$.

2. $f(-a) = -f(a)$ *for all* $a \in R$.

3. If S *a subring of* R, *then* $f(S)$ *is a subring of* R'.

4. If S' *is a subring of* R', *then* $f^{-1}(S')$ *is a subring of* R.

5. If I *is an ideal of* R *and* f *is* **onto**, *then* $f(I)$ *is an ideal of* R'.

6. If I' *is an ideal of* R', *then* $f^{-1}(I')$ *is an ideal of* R.

7. $Ker\ f$ *is an ideal of* R.

8. $Ker\ f = \{0_R\} \Leftrightarrow f$ *is one-to-one.*

9. If R *is commutative, then* $f(R)$ *is commutative.*

10. Let R *has unity* 1_R**.** *Then* $f(R)$ *has unity* $f(1_R)$.

11. If R *has unity* 1_R *and* f *is* **onto.** *Then* R' *has unity* $f(1_R) = 1_{R'}$.

12. If R *has unity* 1_R *and* f *is* **onto.** *If* $a \in R$ *is a unit, then* $f(a)$ *is a unit* R' *and* $f(a)^{-1} = f(a^{-1})$.

13. If $a \in R$ *is an idempotent, then* $f(a) \in R'$ *is an idempotent.*

Examples:

1. Let $f : R \longrightarrow R', f(a) = 0_{R'}$ for all $a \in R$.Then f is a (zero)homomorphism with $Kerf = R$.

2. Let f be an identity map. Then f is an isomorphism with $Kerf = \{0_R\}$.

3. $(\mathbb{Z}, +) \simeq (2\mathbb{Z}, +)$, but $(\mathbb{Z}, +, .) \not\simeq (2\mathbb{Z}, +, .)$ .Since for the function $f : \mathbb{Z} \rightarrow 2\mathbb{Z}$ defined by $f(a) = 2a$, $f(ab) = 2ab \neq f(a) f(b) = 4ab$.

Theorem (Natural Homomorphism)

Let R be a ring and I be an ideal of R. Then the function $\gamma : R \to R/I$ defined by $\gamma(r) = r + I$ is an epimorphism with $Ker\gamma = I$. The homomorphism γ is called the **natural homomorphism** *of R onto R/I.*

- The function $\gamma : \mathbb{Z} \to \mathbb{Z}/\langle n \rangle$ defined by $\gamma(a) = a + \langle n \rangle$ for all $a \in \mathbb{Z}_n$, is the natural homomorphism of $\mathbb{Z}$ onto $\mathbb{Z}/\langle n \rangle$.

Isomorphism Theorems

The following theorem also known as **the fundamental homomorphism theorem**.

Theorem (First Isomorphism Theorem)

Let $f : R \to R'$ be a homomorphism with $Kerf = I$. Then the function $\mu : R/I \to f(R)$ defined by $\mu(r + I) = f(r)$ is an isomorphism; i.e. $R/Kerf \simeq f(R)$. Moreover, if $\gamma : R \to R/I$ is the natural homomorphism, then $f(r) = \mu\gamma(r)$, for each $r \in R$.

- Let $f : \mathbb{Z} \to \mathbb{Z}_n$, $f(a) = \bar{a}$ for all $a \in \mathbb{Z}$. Then f is an epimorphism with $Kerf = n\mathbb{Z} = \langle n \rangle$. Then $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$. Moreover, let $\gamma : \mathbb{Z} \to \mathbb{Z}/\langle n \rangle$, $\gamma(a) = a + \langle n \rangle$ for all $a \in \mathbb{Z}$ is the natural homomorphism, then we have $f = \mu \circ \gamma$, where $\mu : \mathbb{Z}/\langle n \rangle \to \mathbb{Z}_n$ is the isomorphism such that $\mu(a + \langle n \rangle) = \bar{a}$.

Isomorphism Theorems

Theorem (Second Isomorphism Theorem)

Let I and J be ideals of R. Then

$$I/\left(I\cap J\right)\simeq\left(I+J\right)/J$$

Example: Let $I=6\mathbb{Z}$ and $J=10\mathbb{Z}$ be ideals of $\mathbb{Z}$.

$$\begin{aligned} I+J &= 2\mathbb{Z} \\ &\Rightarrow \left(I+J\right)/J=\{0+10\mathbb{Z},2+10\mathbb{Z},4+10\mathbb{Z},6+10\mathbb{Z},8+10\mathbb{Z}\} \\ I\cap J &= 30\mathbb{Z} \\ &\Rightarrow I/\left(I\cap J\right)=\{0+30\mathbb{Z},6+30\mathbb{Z},12+30\mathbb{Z},18+30\mathbb{Z},24+30\mathbb{Z} \end{aligned}$$

Theorem (Third Isomorphism Theorem)

Let I and J be ideals of R such that $I \subseteq J$. Then

$$(R/I)\,/\,(J/I) \simeq R/J.$$

Example: $(\mathbb{Z}/12\mathbb{Z})\,/\,(3\mathbb{Z}/12\mathbb{Z}) \simeq \mathbb{Z}/3\mathbb{Z}$.

Isomorphism Theorems

Theorem (Correspondence Theorem)

Let $f : R \to R'$ be an epimorphism. Then there is a one-to-one correspondence between the ideals of R containing $\mathrm{Ker}f$ and the ideals of R'. That is, if I is an ideal of R containing $\mathrm{Ker}f$, then $f(I)$ is the corresponding ideal of R' and if I' is an ideal of R', then $f^{-1}(I') = \{x \in R \mid f(x) \in I'\}$ is the corresponding ideal of R.

- By Correspondence Theorem, there is a one-to-one correspondence between the ideals of R containing I and the ideals of the quotient ring R/I.

Exercises

1. All ideals of $\mathbb{Z}_{12}$ are $\langle\overline{0}\rangle, \langle\overline{1}\rangle, \langle\overline{2}\rangle, \langle\overline{3}\rangle, \langle\overline{4}\rangle, \langle\overline{6}\rangle$. Then

$$\begin{aligned} \mathbb{Z}_{12}/\langle\overline{0}\rangle &\simeq \mathbb{Z}_{12},\ \mathbb{Z}_{12}/\langle\overline{1}\rangle \simeq \{\overline{0}\},\ \mathbb{Z}_{12}/\langle\overline{2}\rangle \simeq \mathbb{Z}_2, \\ \mathbb{Z}_{12}/\langle\overline{3}\rangle &\simeq \mathbb{Z}_3,\ \mathbb{Z}_{12}/\langle\overline{4}\rangle \simeq \mathbb{Z}_4,\ \mathbb{Z}_{12}/\langle\overline{6}\rangle \simeq \mathbb{Z}_6. \end{aligned}$$

2. $2\mathbb{Z}/8\mathbb{Z} \not\simeq \mathbb{Z}_4$. Since $\mathbb{Z}_4$ has unity, but $2\mathbb{Z}/8\mathbb{Z}$ not.

3. $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\simeq \mathbb{Z}_4$, since their characteristics are different. (Isomorphic rings have the same characteristic).

4. $\mathbb{Z}_4 \times \mathbb{Z}_4 \not\simeq \mathbb{Z}_{16}$, since for $f : \mathbb{Z}_{16} \to \mathbb{Z}_4 \times \mathbb{Z}_4$, no element in $\mathbb{Z}_4 \times \mathbb{Z}_4$ has additive order 16. (Isomorphic rings preserve the additive order).