Lecture 8: Prime and Maximal Ideals

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Prime Ideals

One of the important aims of this chapter is to characterize the ideals by the factor rings of these ideals. To do this, first define the prime and maximal ideals.

Definition

Let R be a commutative ring and P be a proper ideal of R. P is a **prime ideal** of R if

$$ab \in P \Rightarrow a \in P \ or \ b \in P, \text{ for } a, b \in R.$$

An alternative definition can be given as

$$P \overset{prime}{<} R \Leftrightarrow \ a \notin P \text{ and } b \notin P \Rightarrow ab \notin P.$$

Examples:

1. $\langle p \rangle = p\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$.

2. $12\mathbb{Z}$ is not a prime ideal of $\mathbb{Z}$. Since $3.8 = 24 \in \langle 12 \rangle$ but $3 \notin \langle 12 \rangle$ and $8 \notin \langle 12 \rangle$.

3. $2\mathbb{Z} \times 3\mathbb{Z}$ is not a prime ideal of $\mathbb{Z} \times \mathbb{Z}$. Since $(2,1)(1,3) = (2,3) \in 2\mathbb{Z} \times 3\mathbb{Z}$ but $(2,1) \notin 2\mathbb{Z} \times 3\mathbb{Z}$ and $(1,3) \notin 2\mathbb{Z} \times 3\mathbb{Z}$.

Characterization of prime ideals

Theorem

Let R be a commutative ring with unity and P be a proper ideal of R. Then

$$P \text{ is a prime ideal of } R \Leftrightarrow R/P \text{ is an integral domain.}$$

Example: Since $(\mathbb{Z} \times \mathbb{Z}) \,/\, (\mathbb{Z} \times \{0\}) \simeq \mathbb{Z}$, $\mathbb{Z} \times \{0\}$ is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$.

Maximal Ideals

Definition

Let R be a ring and M be a proper ideal of R. M is a **maximal ideal** of R if there is no proper ideal I of R properly containing M.
That is, M is a **maximal ideal** of $R \Leftrightarrow$ if I is an ideal of R such that $M \subseteq I \subseteq R$, then either $I = M$ or $I = R$.

Existence of maximal ideals

Theorem

Let R be a commutative ring with unity. Then every proper ideal of R is contained in a maximal ideal of R.

Corollary

1. *For a maximal ideal M of R, $a \in M \Leftrightarrow a \notin U(R)$.*
2. *R has at least one maximal ideal.*

Characterization of maximal ideals

Theorem

Let R be a commutative ring with unity and M be a proper ideal of R. Then

$$M \text{ is a maximal ideal of } R \Leftrightarrow R/M \text{ is a field.}$$

Example: $4\mathbb{Z} = \langle 4 \rangle$ is a maximal ideal of $2\mathbb{Z}$, but not prime. Since $2.2 \in \langle 4 \rangle$ but $2 \notin \langle 4 \rangle$.

Prime and Maximal Ideals

Theorem

Let R be a commutative ring with unity.

$$M \text{ is a maximal ideal of } R \Rightarrow M \text{ is a prime ideal of } R.$$

Example:

$$\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n \text{ is a field} \Leftrightarrow n \text{ is a prime,}$$

Thus, the maximal ideals of $\mathbb{Z}$ are $p\mathbb{Z}$, for prime p. Thus the ideals $p\mathbb{Z}$ are also prime.

Remark: The converse of this fact may not be true. For example, Since $\mathbb{Z}/\langle 0\rangle \simeq \mathbb{Z}$ is an integral domain, the zero ideal $\langle 0\rangle$ is a prime ideal of $\mathbb{Z}$, but not maximal.

Prime and Maximal Ideals

Theorem

Let R be a principle ideal domain (PID).

$$P \text{ is a prime ideal of } R \Leftrightarrow P \text{ is a maximal ideal of } R.$$

Prime and Maximal Ideals

Example: Find the maximal ideals of $\mathbb{Z}_6$ are $\{\overline{0},\overline{2},\overline{4}\}$ and $\{\overline{0},\overline{3}\}$.

Since

$$\begin{aligned} f:\mathbb{Z} &\rightarrow \mathbb{Z}_6 \\ n &\longrightarrow \overline{n} \end{aligned}$$

is an epimorphism with $Kerf = 6\mathbb{Z}$, the ideals of $\mathbb{Z}_6$ are $f(I)$ such that $6\mathbb{Z} \subseteq I$.

The only ideals of $\mathbb{Z}$ contains $6\mathbb{Z}$ are $\mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}$, and $6\mathbb{Z}$. Then $f(\mathbb{Z}) = \mathbb{Z}_6, f(2\mathbb{Z}) = \{\overline{0},\overline{2},\overline{4}\}, f(3\mathbb{Z}) = \{\overline{0},\overline{3}\}$, and $f(6\mathbb{Z}) = \{\overline{0}\}$.