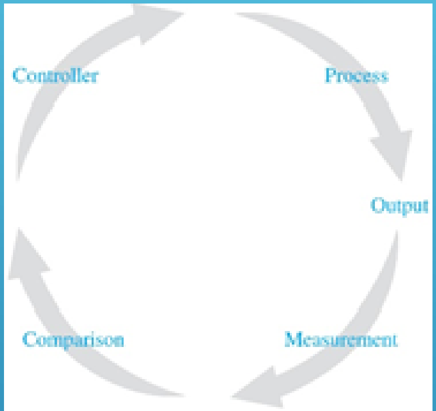
CONTROL SYSTEMS



Doç. Dr. Murat Efe

WEEK 2

Laplace transform of *Exponential Function*

$$f(t) = 0 \quad \text{for } t < 0$$
$$f(t) = Ae^{-\alpha t} \quad \text{for } t \geq 0$$
where α and A are constants

$$L\{Ae^{-\alpha t}\} = \int_0^\infty Ae^{-\alpha t} e^{-st} dt = \int_0^\infty Ae^{-(s+\alpha)t} dt = \frac{A}{s+\alpha}$$

The abscissa of convergence: $s > -\alpha$

Exponential function produces a pole in the complex plane

Laplace transform of *Step Function,* 1(t)

$$f(t) = 0 \quad \text{for } t < 0$$
$$f(t) = A \quad \text{for } t > 0$$
where A is a constant

$$L\{A\} = \int_0^\infty A e^{-st} dt = \frac{A}{s}$$

The abscissa of convergence: $s > 0$

Step function produces a pole at the origin of the complex plane

Laplace transform of *Ramp Function*

$$f(t) = 0 \quad \text{for } t < 0$$
$$f(t) = At \quad \text{for } t \geq 0$$

$$L\{At\} = \int_0^\infty Ate^{-st}\,dt = At\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty A\frac{e^{-st}}{-s}\,dt$$

$$= \frac{A}{s}\int_0^\infty e^{-st}\,dt = -\frac{A}{s^2}e^{-st}\Bigg|_0^\infty = \frac{A}{s^2}$$

Ramp function produces double poles at the origin of the complex plane

Laplace transform of *Sinusoidal Function*

$$f(t) = 0 \qquad \text{for } t < 0$$
$$f(t) = A\sin\omega t \qquad \text{for } t \geq 0$$
where A and ω are constants

$$L\{A\sin\omega t\} = \int_0^\infty A\sin\omega t e^{-st}dt = \int_0^\infty \frac{A}{2j}(e^{j\omega t} - e^{-j\omega t})e^{-st}dt$$

$$= \frac{A}{2j}\frac{1}{s - j\omega} - \frac{A}{2j}\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2} \quad \text{Similarly}$$

$$L\{A\cos\omega t\} = \frac{As}{s^2 + \omega^2}$$

Sinusoidal functions produce poles on the imaginary ($j\omega$) axis

Several Properties of Laplace Transform
&
Laplace Transforms of Important Functions

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

1. Linearity

$$L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\}$$

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty (\alpha f(t) + \beta g(t)) e^{-st} \mathrm{d}t \\
&= \int_0^\infty \alpha f(t) e^{-st} \mathrm{d}t + \int_0^\infty \beta g(t) e^{-st} \mathrm{d}t \\
&= \alpha \int_0^\infty f(t) e^{-st} \mathrm{d}t + \beta \int_0^\infty g(t) e^{-st} \mathrm{d}t \\
&= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}
\end{aligned}$$

2. Time Shift (Delay, Advance)

$$\boxed{L\{f(t-\tau)\} = e^{-\tau s}F(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\begin{aligned}
\mathcal{L}\{f(t-a)\} &= \int_0^\infty f(t-a)e^{-st}\,\mathrm{d}t \\
&= \int_{-a}^\infty f(y)e^{-s(a+y)}\,\mathrm{d}y \\
&= \int_0^\infty f(y)e^{-s(a+y)}\,\mathrm{d}y \\
&= e^{-as}\int_0^\infty f(y)e^{-sy}\,\mathrm{d}y
\end{aligned}$$

3. Multiplication by e^{-at}

$$\boxed{L\{e^{-at} f(t)\} = F(s+a)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\begin{aligned}\mathcal{L}\{e^{-at} f(t)\} &= \int_0^\infty e^{-at} f(t) e^{-st} \mathrm{d}t \\ &= \int_0^\infty f(t) e^{-(s+a)t} \mathrm{d}t\end{aligned}$$

4. Change of Time Scale $a > 0$

$$\boxed{L\{f(t/a)\} = aF(as)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\begin{aligned}
\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} &= \int_0^\infty f\left(\frac{t}{a}\right) e^{-st}\mathrm{d}t \\
&= \int_0^\infty f(y)e^{-say}a\mathrm{d}y \\
&= a\int_0^\infty f(y)e^{-(as)y}\mathrm{d}y \\
&= aF(as)
\end{aligned}$$

5. Real Differentiation

$$\boxed{L\{\frac{d}{dt}f(t)\} = sF(s) - f(0)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\int_0^\infty f(t)e^{-st}\mathrm{d}t = f(t)\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{e^{-st}}{-s}\mathrm{d}t$$

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}e^{-st}\mathrm{d}t$$

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0)$$

6. Real Integration

$$\boxed{L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

If $f(t)$ is of exp. order

$$\mathcal{L}\left\{\int_0^t f(y)\mathrm{d}y\right\} = \int_0^\infty \left(\int_0^t f(y)\mathrm{d}y\right) e^{-st}\mathrm{d}t$$

Apply integration by parts

$$\int_0^\infty \left(\int_0^t f(y)\mathrm{d}y\right) e^{-st}\mathrm{d}t = \left(\int_0^t f(y)\mathrm{d}y\right) \frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}\mathrm{d}t$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}\mathrm{d}t$$

$$\mathcal{L}\left\{\int_0^t f(y)\mathrm{d}y\right\} = \frac{1}{s}F(s)$$

7. Multiplication by t

$$\boxed{L\{tf(t)\} = \frac{d}{ds}F(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

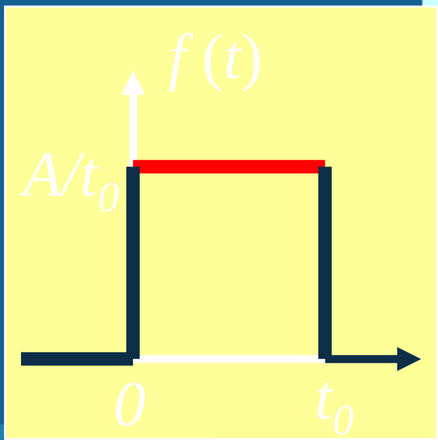
$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

8. Laplace Transform of a Pulse

$$f(t) = \frac{A}{t_0} \quad \text{for } 0 < t < t_0$$

$$f(t) = 0 \quad \text{for } t < 0, t_0 < t$$

$$f(t) = \frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)$$



$$L\{f(t)\} = L\left\{\frac{A}{t_0} 1(t) - \frac{A}{t_0} 1(t - t_0)\right\}$$

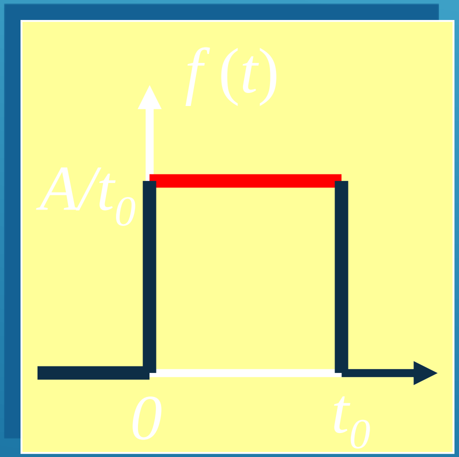
$$= \frac{A}{t_0}\left(L\{1(t)\} - L\{1(t - t_0)\}\right)$$

$$= \frac{A}{t_0} L\{1(t)\}\left(1 - e^{-st_0}\right)$$

$$= \frac{A}{t_0}\frac{1}{s}\left(1 - e^{-st_0}\right)$$

9. Laplace Transform of Impulse Function

$$L\left\{\lim_{t_0\to 0} f(t)\right\} = \lim_{t_0\to 0}\left(\frac{A}{t_0}\frac{1}{s}\left(1-e^{-st_0}\right)\right) = A$$



10. Final Value Theorem

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

This theorem can be applicable if

- $f(t)$ settles down to a constant limit
- sF(s) has no poles on the imaginary axis, this obviously means oscillations in $f(t)$
- sF(s) has no poles on the right half s-plane

11. Initial Value Theorem

$$\boxed{f(0+) = \lim_{s\to\infty} sF(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

This Theorem can be applicable if

- $f(t)$ and $df(t)/dt$ are both Laplace transformable
- The limit on the right hand side exists

12. Laplace Transform of Convolution

$$L\{f(t) * g(t)\} = F(s)G(s)$$

where

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

and by duality

$$L\{f(t)g(t)\} = F(s) * G(s)$$

$$t - y := p$$

$$\begin{aligned}
\int_0^\infty \int_0^\infty f(y)g(t-y)e^{-st}\mathrm{d}y\mathrm{d}t &= \int_0^\infty \int_0^\infty f(y)g(t-y)e^{-st}\mathrm{d}t\mathrm{d}y \\
&= \int_0^\infty \int_{-y}^\infty f(y)g(p)e^{-s(p+y)}\mathrm{d}p\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty f(y)g(p)e^{-s(p+y)}\mathrm{d}y\mathrm{d}p \\
&= \int_0^\infty f(y)e^{-sy}\mathrm{d}y \int_0^\infty g(p)e^{-sp}\mathrm{d}p \\
&= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}
\end{aligned}$$

Inverse Laplace Transform

Typical Inversion Methods

- Use of inversion integral
  Complicated and generally takes long time
- Use of table (Textbook pp.22-23)
  Easiest way but you may not always be able to find what you are looking for in the table explicitly

We will take a look at Partial Fraction Expansion

Partial Fraction Expansion

Consider

$$F(s)=\frac{B(s)}{A(s)}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where m<n.

- If m=n, then find out the constant term and separately write in the expansion, then invert.
- If m>n, then find out the polynomial in s, and write and invert it separately.
- $-z_i$'s are zeros and $-p_i$'s are poles.
- Poles and zeros may be complex numbers as well

If m<n, the expression

$$\boxed{F(s)=\frac{B(s)}{A(s)}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}}$$

can be expanded as

$$\boxed{F(s)=\frac{B(s)}{A(s)}=\frac{a_1}{s+p_1}+\frac{a_2}{s+p_2}+\cdots+\frac{a_n}{s+p_n}}$$

where

$$\boxed{a_i=\left[(s+p_i)F(s)\right]_{s=-p_i}}$$

Consider

$$F(s)=\frac{B(s)}{A(s)}=\frac{B(s)}{\left(\prod_{i=1}^{m}(s+p_i)\right)(s+\alpha)^q}$$

where, deg A<q+m. This expression can be expanded as

$$F(s)=\frac{B(s)}{A(s)}=\sum_{i=1}^{m}\frac{a_i}{s+p_i}+\sum_{j=1}^{q}\frac{c_j}{(s+\alpha)^j}$$

with

$$c_{q-k}=\left[\frac{1}{k!}\frac{d^k}{ds^k}\left((s+\alpha)^q F(s)\right)\right]_{s=-\alpha}$$

$k=0,1,\ldots,q-1$

Example (2.3 from book)

Find the inverse Laplace transform of

$$\frac{1}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$

Example (2.4 from book)

Find the inverse Laplace transform of

$$\frac{1}{(s^2+1)(s+2)} = \frac{R_1}{s+j} + \frac{R_2}{s-j} + \frac{R_3}{s+2}$$

$$R_1 = ((s+j)F(s))|_{s=-j} = -\frac{1}{10} + j\frac{1}{5}$$

$$R_2 = ((s-j)F(s))|_{s=j} = -\frac{1}{10} - j\frac{1}{5} = \overline{R_1}$$

$$R_3 = ((s+2)F(s))|_{s=-2} = \frac{1}{5}$$

$$f(t) = \left(\frac{1}{5}e^{-2t} + \left(-\frac{1}{10} + j\frac{1}{5}\right)e^{-jt} + \left(-\frac{1}{10} - j\frac{1}{5}\right)e^{jt}\right)1(t)$$

$$e^{jt} + e^{-jt} = 2\cos(t)$$

$$e^{jt} - e^{-jt} = 2j\sin(t)$$

$$f(t) = \frac{1}{5}(e^{-2t} - \cos(t) + 2\sin(t))1(t)$$

An Example

Find the inverse Laplace transform of

$$F(s) = \frac{B(s)}{A(s)} = \frac{s+3}{(s+1)(s+2)^3}$$

Solution: Rewrite it as

$$F(s) = \frac{a_1}{s+1} + \frac{c_1}{s+2} + \frac{c_2}{(s+2)^2} + \frac{c_3}{(s+2)^3}$$

$$a_1 = \left[(s+1)F(s)\right]_{s=-1} = 2$$

$$c_3 = \left[(s+2)^3 F(s)\right]_{s=-2} = -1$$

$$c_2 = \frac{d}{ds}\left[(s+2)^3 F(s)\right]_{s=-2} = -2$$

$$c_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+2)^3 F(s)\right]_{s=-2} = -2$$

$$F(s) = \frac{2}{s+1} + \frac{-2}{s+2} + \frac{-2}{(s+2)^2} + \frac{-1}{(s+2)^3}$$

$$L\left\{\frac{t^{n-1}e^{-at}}{(n-1)!}\right\} = \frac{1}{(s+a)^n} \text{ where } n = 1,2,3,\ldots$$

$$f(t) = 2e^{-t} - 2e^{-2t} - 2te^{-2t} - \frac{1}{2}t^2e^{-2t}$$

$$f(t) = 2e^{-t} - (1/2)(4 + 4t + t^2)e^{-2t}$$

Example (2.5 from book)

Find the inverse Laplace transform of

$$F(s) = \frac{(s+1)^3}{(s+2)^4}$$

$$F(s)=\frac{R_1}{s+2}+\frac{R_2}{(s+2)^2}+\frac{R_3}{(s+2)^3}+\frac{R_4}{(s+2)^4}$$

$$R_4=\left((s+2)^4F(s)\right)|_{s=-2}=-1$$

$$R_3=\left(\frac{\mathrm{d}}{\mathrm{d}s}(s+2)^4F(s)\right)\bigg|_{s=-2}=3$$

$$R_2=\frac{1}{2!}\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}(s+2)^4F(s)\right)\bigg|_{s=-2}=-3$$

$$R_1=\frac{1}{3!}\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}(s+2)^4F(s)\right)\bigg|_{s=-2}=1$$

$$F(s)=\frac{1}{s+2}+\frac{-3}{(s+2)^2}+\frac{3}{(s+2)^3}+\frac{-1}{(s+2)^4}$$

$$\mathcal{L}\{e^{-2t}1(t)\} = \frac{1}{s+2} = Q(s)$$

$$\mathcal{L}\{te^{-2t}1(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}Q(s) = \frac{1}{(s+2)^2}$$

$$\mathcal{L}\{t^2e^{-2t}1(t)\} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}Q(s) = \frac{2}{(s+2)^3}$$

$$\mathcal{L}\{t^3e^{-2t}1(t)\} = -\frac{\mathrm{d}^3}{\mathrm{d}s^3}Q(s) = \frac{6}{(s+2)^4}$$

$$f(t) = -\frac{1}{6}\left(t^3 - 9t^2 + 18t - 6\right)e^{-2t}1(t)$$

$f(t)$	$F(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{t^{n-1}e^{-at}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$

$f(t)$	$F(s)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\frac{1-e^{-at}}{a}$	$\frac{1}{s(s+a)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

$f(t)$	$F(s)$
$\frac{be^{-bt}-ae^{-at}}{b-a}$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{ab}\left(1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right)$	$\frac{s}{s(s+a)(s+b)}$
$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{(s+a)}{(s+a)^2+\omega^2}$

P-1 Review of Linear Algebra

Inner (Dot) Product of Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then } x \cdot y = x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i$$

Multiplication of Two Matrices

$$C = AB$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k} a_{ik} b_{kj}$$

Determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} +$$

$$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{32}) - a_{12}(a_{11}a_{33} - a_{13}a_{31}) +$$

$$a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{12} & a_{12} & a_{13} \\ a_{21}+a_{22} & a_{22} & a_{23} \\ a_{31}+a_{32} & a_{32} & a_{33} \end{vmatrix}$$

Given a determinant, summing two rows and writing the result as one of those rows do not change the value of the determinant.

Similarly, summing two columns and using the result as one of those columns do not change the value of the determinant.

Eigenvalues and Eigenvectors

$$\left|\lambda I - A\right| = 0$$

$$Av = \lambda v$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix}$$

Characteristic Polynomial

$$\left|\lambda I - A\right| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0$$

*Note that a polynomial is said to be **monic** if the coefficient of the highest order term is equal to unity*

Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic polynomial

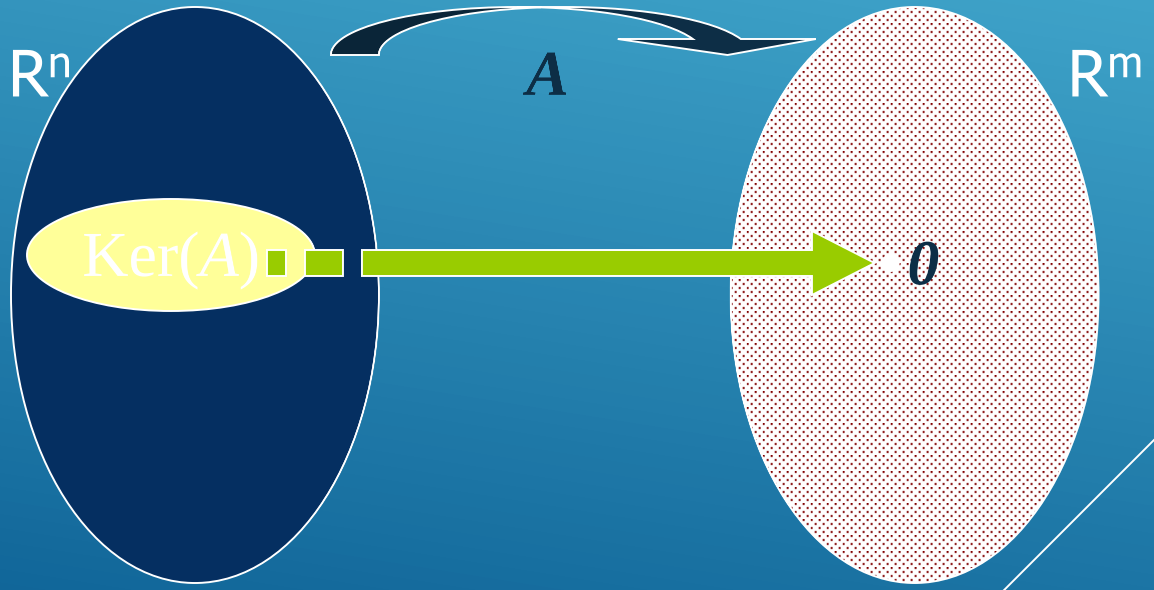
$$A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0$$

Kernel and Image

$$A : R^n \to R^m$$

$$\mathrm{Ker}(A) = \mathrm{Null}(A) := \{x \in R^n : Ax = 0\}$$

$$\mathrm{Im}(A) = \mathrm{Range}(A) := \{y \in R^m : y = Ax, x \in R^n\}$$



Kernel and Image

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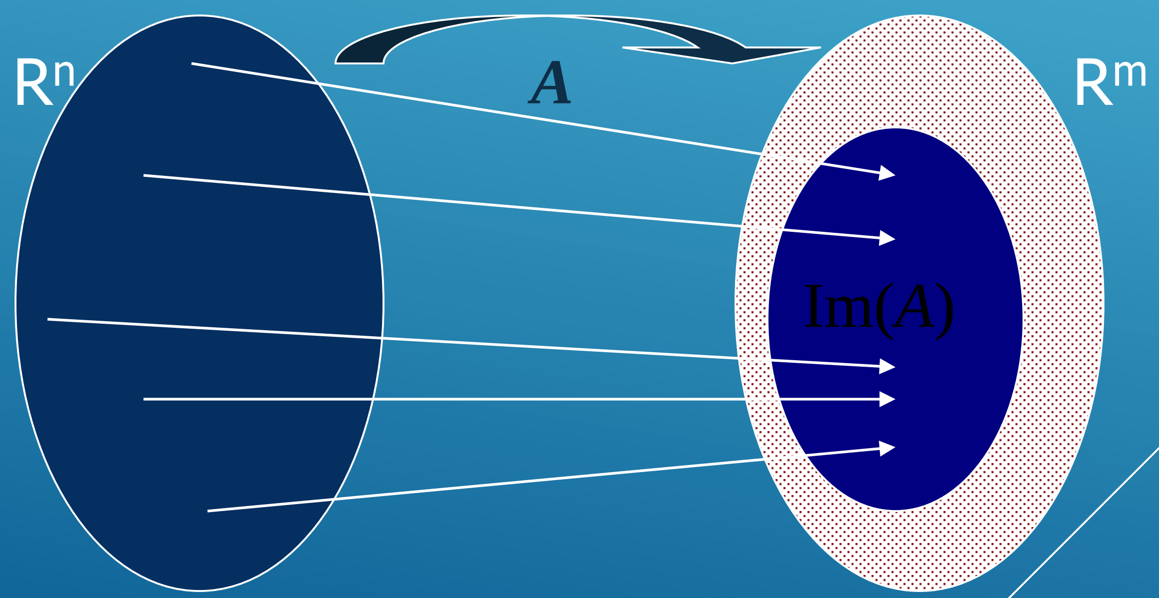

Linear Dependence/Independence

Let $\boxed{x_i \in R^n}$

Set $\boxed{x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k}$

If a set of α_j (**other than all zero**) yields $x=0$, then $\{x_1, x_2, \ldots, x_k\}$ set is said to be linearly dependent otherwise $x_{1 \ldots k}$ are linearly independent