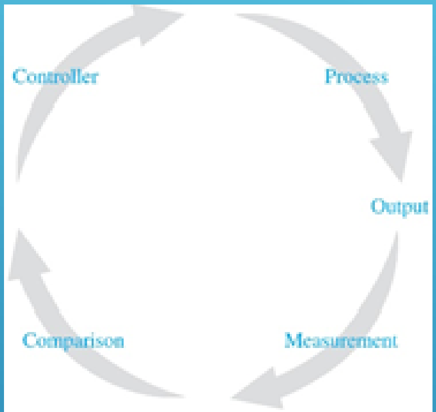
CONTROL SYSTEMS



Doç. Dr. Murat Efe

WEEK 5

This week's agenda

- Concept of Stability
- Stability Analysis of the Closed Loop System by Routh Criterion
- State Space Representation and Stability

P-3 Concept of Stability

What is stability?

- Stability is a property of the system regardless of the signals at the inputs and outputs
- Stability is an underlying requirement in every control system

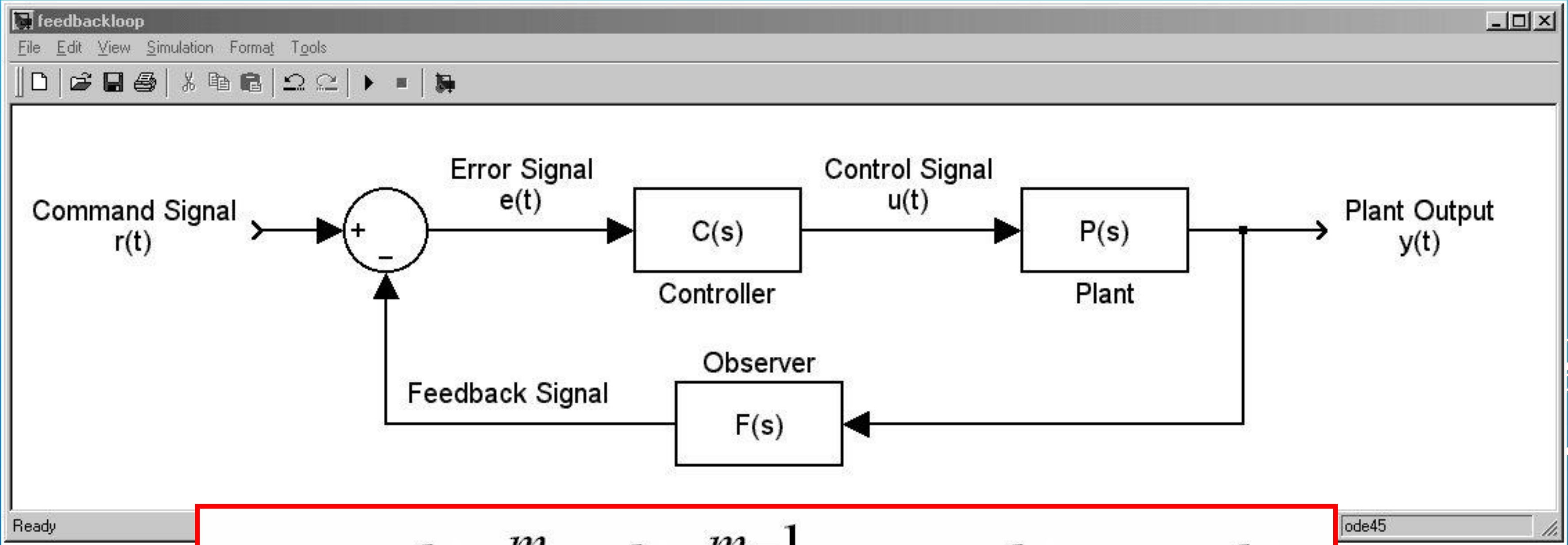
Why do we need to analyze stability?

- An unstable system is potentially dangerous!
- When the power is turned on, the output will increase (decrease/oscillate) indefinitely…
- Eventually this will damage the physical setup

P-3 Stability Analysis of the Closed Loop System by Routh Criterion

Consider the feedback loop

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1+P(s)C(s)F(s)} = T(s)$$



$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

s^n	a_0	a_2	a_4	a_6	$\cdot$	$\cdot$	$\cdot$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdot$	$\cdot$	$\cdot$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdot$	$\cdot$	$\cdot$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdot$	$\cdot$	$\cdot$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$					
$\cdot$	$\cdot$	$\cdot$					
$\cdot$	$\cdot$	$\cdot$					
s^2	e_1	e_2					
s^1	f_1						
s^0	g_1						

T(s) will let you first place these terms

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1+P(s)C(s)F(s)} = T(s)$$

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

ROW #3

Evaluate till the remaining bs are all zero

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

s^n	a_0	a_2	a_4	a_6	.	.	.
s^{n-1}	a_1	a_3	a_5	a_7	.	.	.
s^{n-2}	b_1	b_2	b_3	b_4	.	.	.
s^{n-3}	c_1	c_2	c_3	c_4	.	.	.
s^{n-4}	d_1	d_2	d_3	d_4	.	.	.
.	.	.					
.	.	.					
.	.	.					
s^2	e_1	e_2					
s^1	f_1						
s^0	g_1						

ROW #4

Evaluate till the remaining cs are all zero

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

s^n	a_0	a_2	a_4	a_6	.	.	.
s^{n-1}	a_1	a_3	a_5	a_7	.	.	.
s^{n-2}	b_1	b_2	b_3	b_4	.	.	.
s^{n-3}	c_1	c_2	c_3	c_4	.	.	.
s^{n-4}	d_1	d_2	d_3	d_4	.	.	.
.	.	.					
.	.	.					
.	.	.					
s^2	e_1	e_2					
s^1	f_1						
s^0	g_1						

ROW #5

Evaluate till the remaining bs are all zero

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$d_3 = \frac{c_1 b_4 - b_1 c_4}{c_1}$$

$$\vdots$$

s^n	a_0	a_2	a_4	a_6	$\cdot$	$\cdot$	$\cdot$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdot$	$\cdot$	$\cdot$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdot$	$\cdot$	$\cdot$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdot$	$\cdot$	$\cdot$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$					
$\cdot$	$\cdot$	$\cdot$					
$\cdot$	$\cdot$	$\cdot$					
s^2	e_1	e_2					
s^1	f_1						
s^0	g_1						

[redacted]

Remarks

- Repeat the same pattern till you reach the end i.e. g_1
- The complete array of coefficients is triangular
- Dividing or multiplying any row by a positive number can simplify the calculation without altering the stability conclusion

Routh's stability criterion states that

For

$$T(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

The number of poles on the right hand s-plane is equal to the number of sign changes in the first column of the table

Note that, we only need the signs of the numbers in the first column

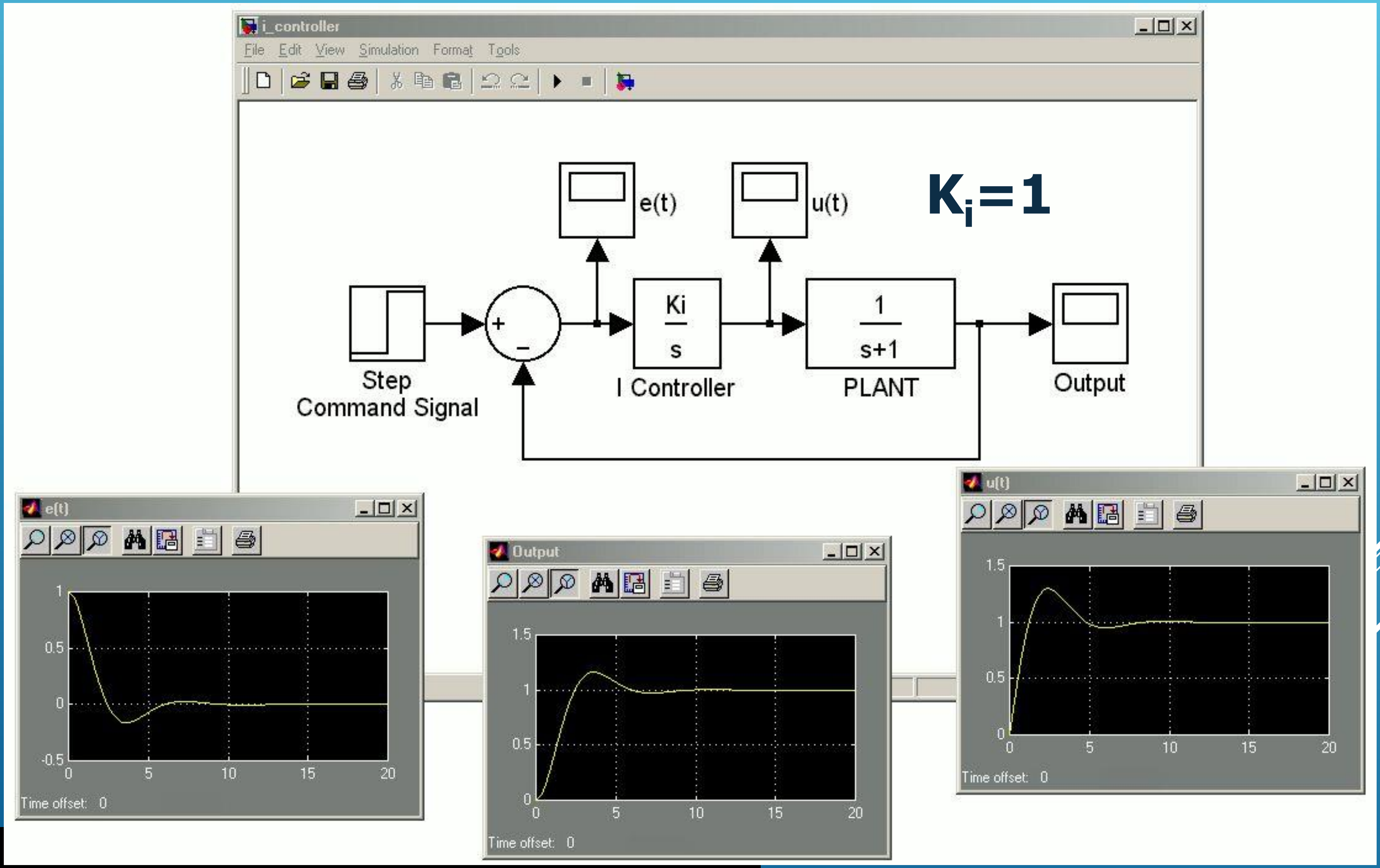
In other words...

s^n	a_0	a_2	a_4	a_6	.	.	.
s^{n-1}	a_1	a_3	a_5	a_7	.	.	.
s^{n-2}	b_1	b_2	b_3	b_4	.	.	.
s^{n-3}	c_1	c_2	c_3	c_4	.	.	.
s^{n-4}	d_1	d_2	d_3	d_4	.	.	.
.	.	.					
.	.						
.	.	.					
s^2	e_1	e_2					
s^1	f_1						
s^0	g_1						

These terms must have the same signs for stability

First Example

Recall that we analyzed the following diagram in I-Controller



First Example

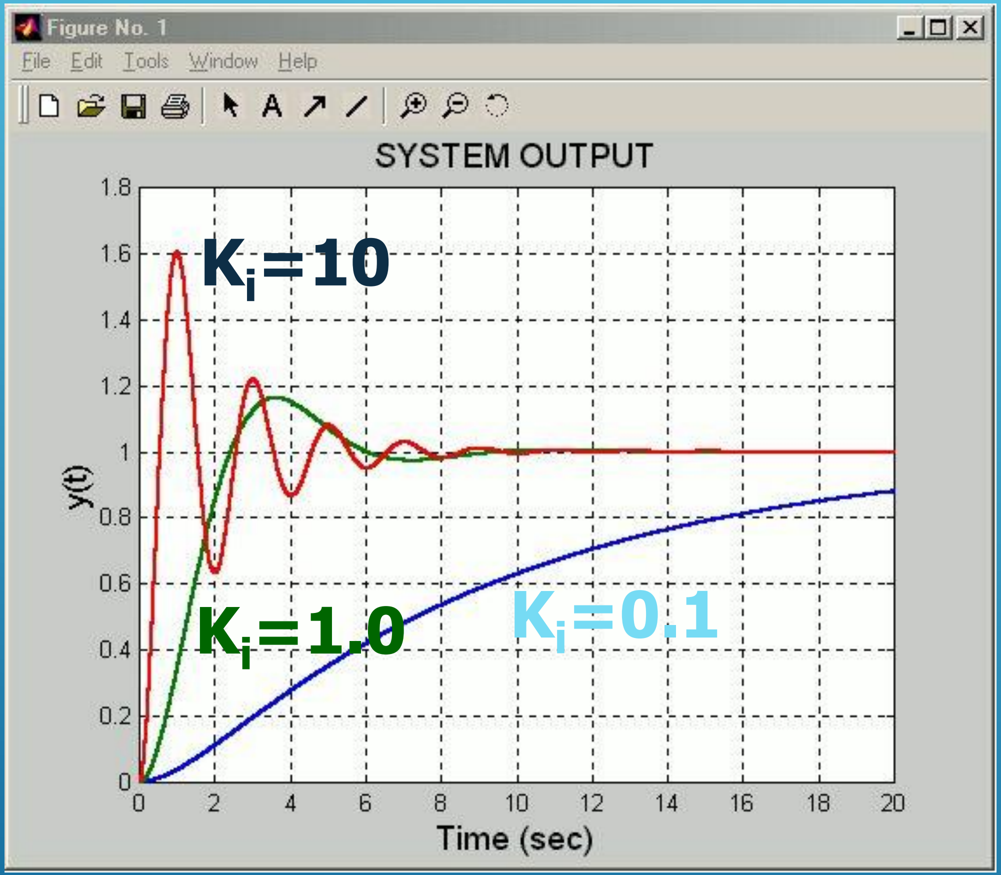
Did we have to choose $K_i=1$? NO!

$$T(s) = \frac{K_i}{s^2 + s + K_i}$$

s^2	1	K_i
s^1	1	
s^0	K_i	

For no sign change in the first column, $K_i>0$ is required. Any positive integral gain would work fine

First Example - System Output



Notice that what they do ultimately are the same, but how they do differ.

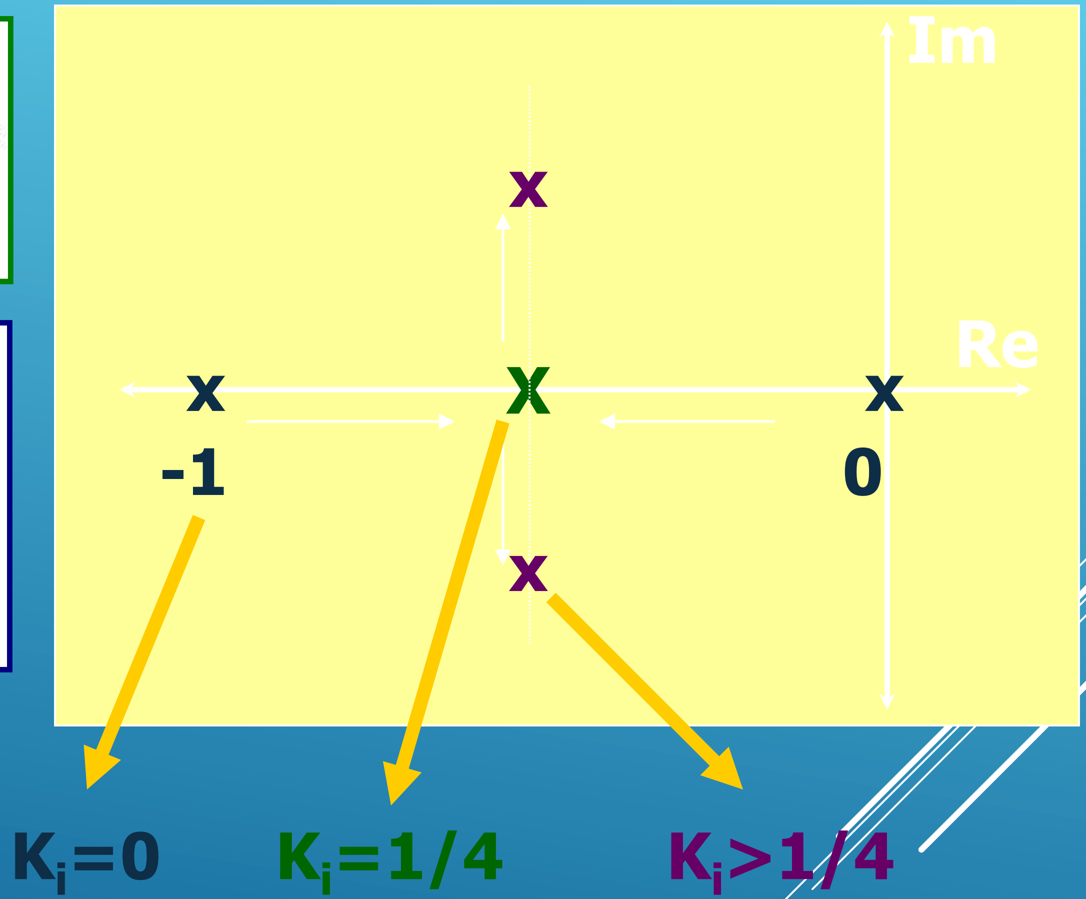
Small $K_i \Rightarrow$ Overdamped (Approaches very slowly)
Large $K_i \Rightarrow$ Underdamped (More quickly but with oscillations)

First Example
Where do the oscillations come from?

$$T(s) = \frac{K_i}{s^2 + s + K_i}$$

$$\Delta = 1 - 4K_i$$

$$s_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - K_i}$$

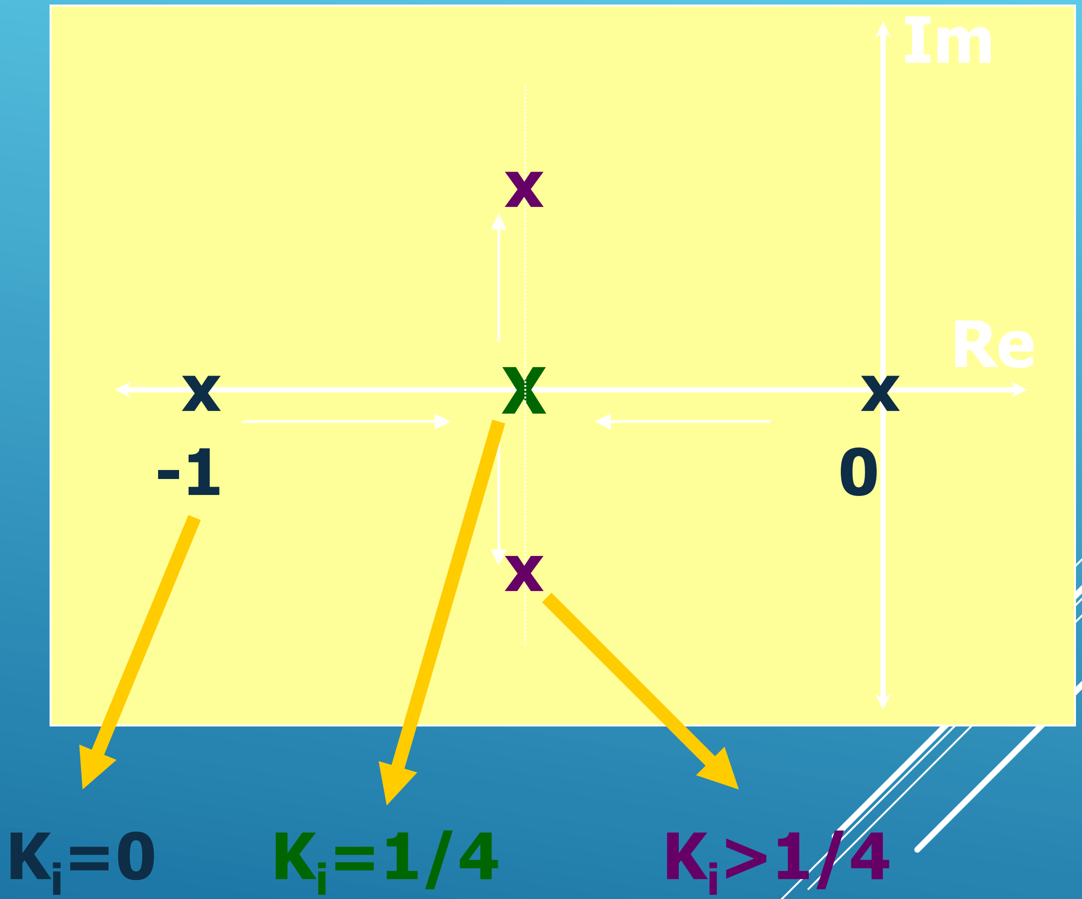


First Example
Where do the oscillations come from?

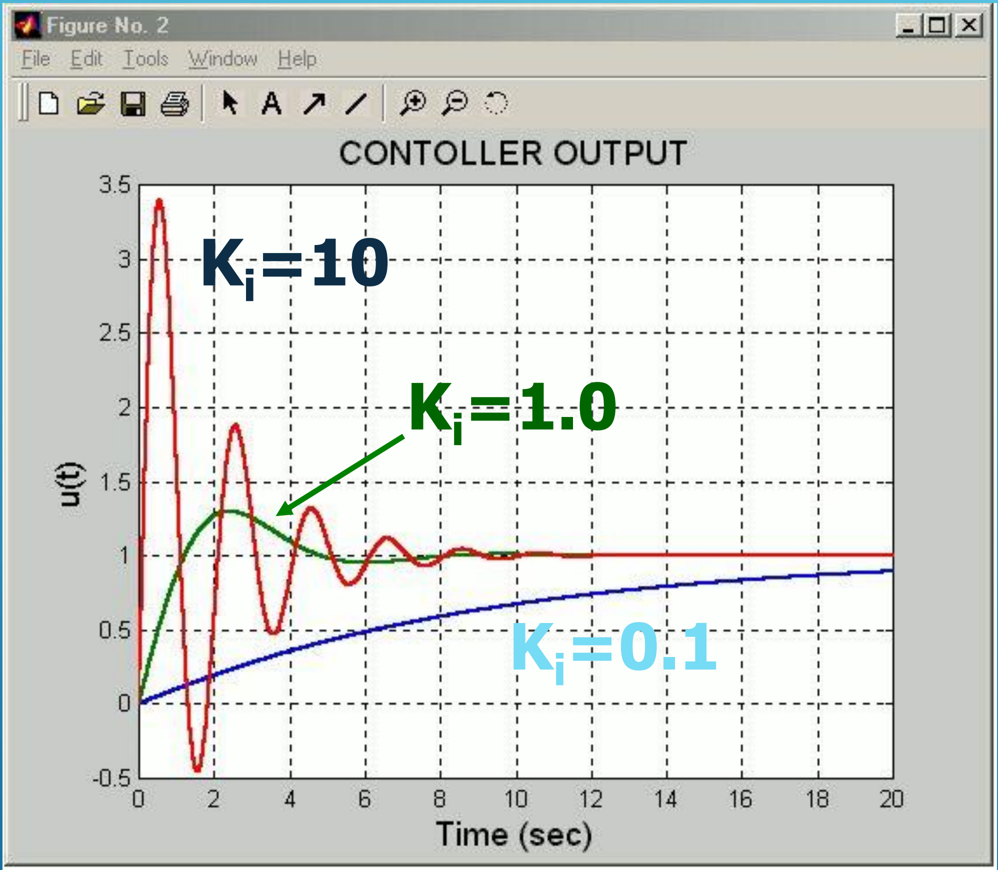
$0<K_i<1/4$
Distinct real poles

$K_i=1/4$
Double poles at $s=-1/2$

$K_i>1/4$
Complex conjugate poles with real parts $-1/2$



First Example - Controller Output



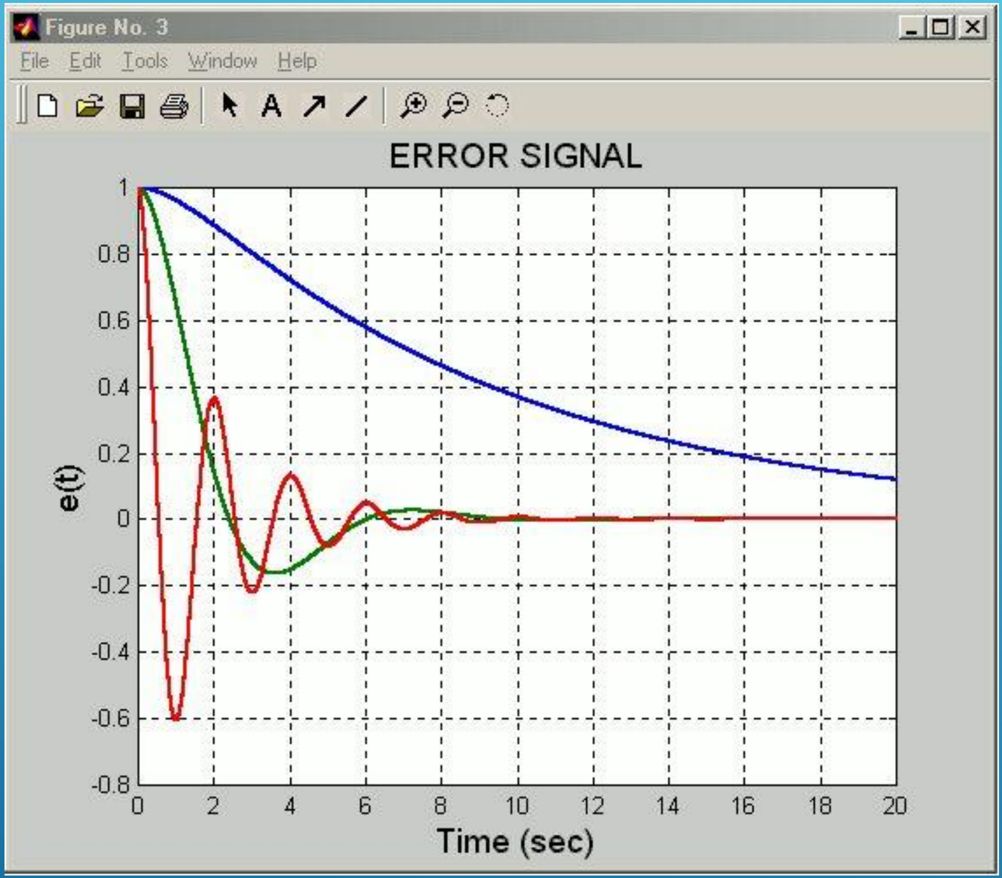
$0<u(t)<1$ for $K_i=0.1$
$0<u(t)<1.3$ for $K_i=1$
$-0.45<u(t)<3.4$ for $K_i=10$

As the controller gain is increased, the range of control signal expands.

- **Can your physical controller provide it?**
- **Is that control signal applicable?**

Small $K_i \Rightarrow$ Overdamped (Approaches very slowly)
Large $K_i \Rightarrow$ Underdamped (More quickly but with oscillations)

First Example - Error Signals



How fast you want the error signal come down to zero?

This signal is the input to the controller. Is that physically applicable to your controller?

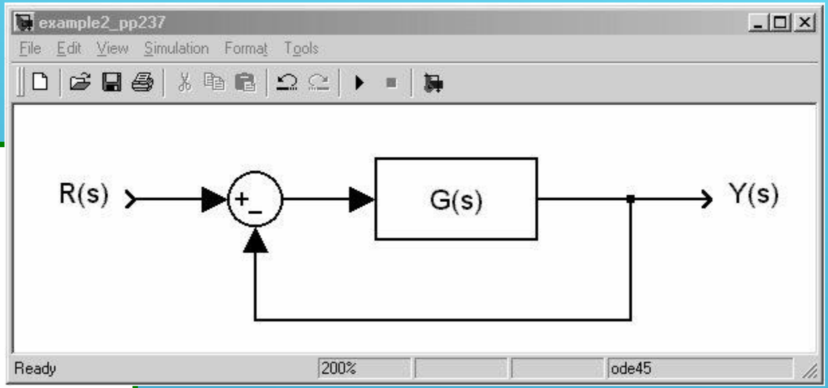
First Example - Remarks

- We learned how to check stability of the closed loop (CL) TF
- A set of controller gains (K_i for this example) can result in stable CL. We analyzed what happens with different values
- We learned what questions to ask in the design phase

Second Example

$$G(s) = \frac{K}{s\left(s^2+s+1\right)(s+2)}$$

$$T(s) = \frac{G}{1+G} = \frac{K}{s\left(s^2+s+1\right)(s+2)+K}$$



Determine the range of K for stability

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Second Example (Textbook pp.237)

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	
s^1	$2-(9/7)K$		
s^0	K		

$$2-(9/7)K > 0$$

$$K > 0$$

$$0 < K < \frac{14}{9}$$

Handling the special cases - Example 1
A zero in the first column

Consider $s^3 + 2s^2 + s + 2 = 0$

s^3	1	1
s^2	2	2
s^1	0	
s^0	?	

Insert ε for 0

ε>0

s^3	1	1
s^2	2	2
s^1	$0 \approx \varepsilon$	
s^0	$2\varepsilon/\varepsilon = 2$	

Handling the special cases - Example 1
A zero in the first column

- No sign change means no roots on the **right half s-plane**
- In this example, two roots were at s=±j

$$s^3 + 2s^2 + s + 2 = 0$$

s^3	1	1
s^2	2	2
s^1	$0 \approx \varepsilon$	
s^0	$2\varepsilon/\varepsilon = 2$	

Handling the special cases - Example 2
A zero in the first column

$$s^3 - 3s + 2 = 0$$

s^3	1	-3
s^2	$0 \approx \varepsilon$	2
s^1	$-3 - 2/\varepsilon$	
s^0	2	

One sign change

One sign change

- **Two sign changes mean two roots on the right half s-plane**

$$s^3 - 3s + 2 = (s-1)^2(s+2) = 0$$

Handling the special cases - Remarks

$\cdot$	
$\cdot$	
s^{k}	positive
s^{k-1}	$0 \approx \varepsilon$
s^{k-2}	positive
$\cdot$	
$\cdot$	

No sign change, i.e. no roots on the right half s-plane

But, there are a pair of imaginary roots

Handling the special cases - Remarks

.	
.	
s^k	positive
s^{k-1}	$0 \approx \varepsilon$
s^{k-2}	negative
.	
.	

or

.	
.	
s^k	negative
s^{k-1}	$0 \approx \varepsilon$
s^{k-2}	positive
.	
.	

One sign change, i.e. there is one root on the right half s-plane from this change