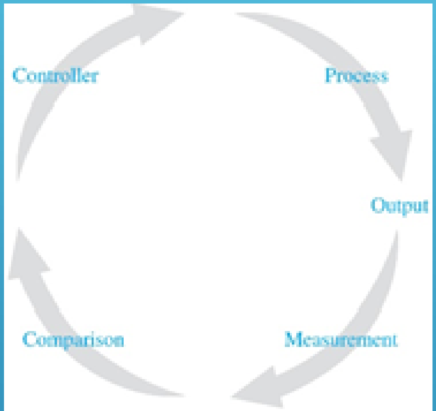
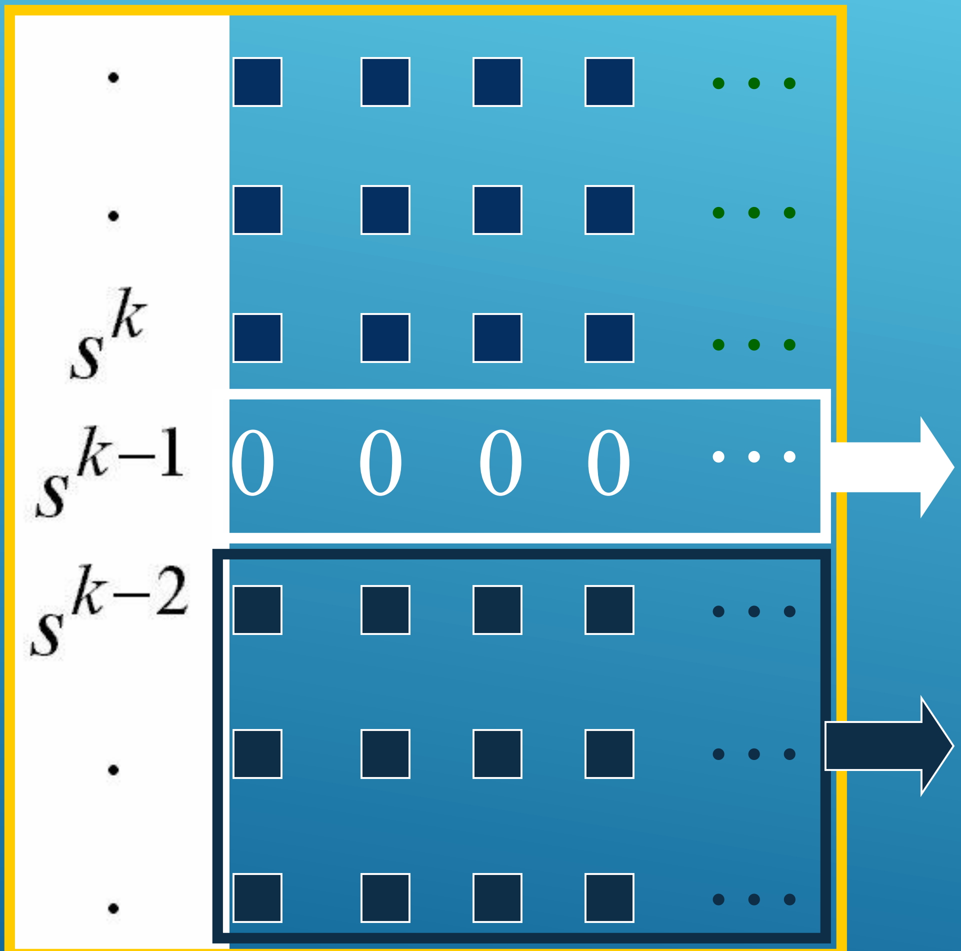
Control Systems



Doç. Dr. Murat Efe

WEEK 6

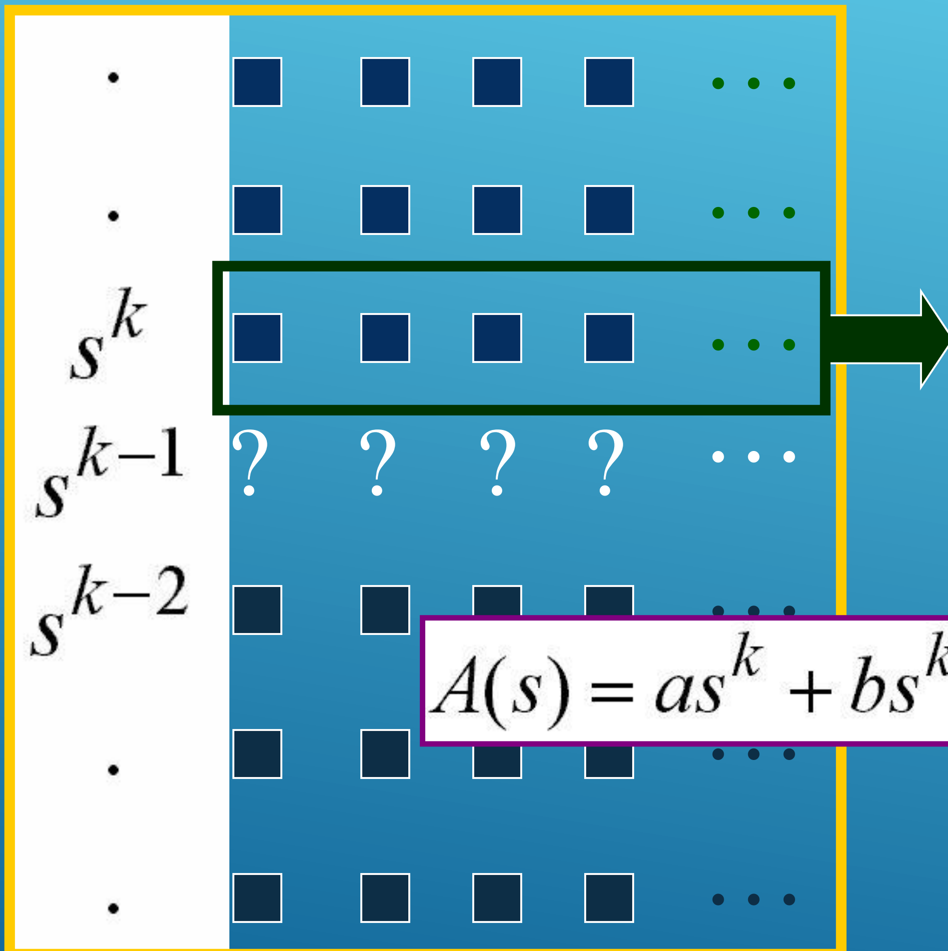
Handling the special cases
A row is entirely zero



This row is entirely zero!

You cannot proceed to calculate these terms!

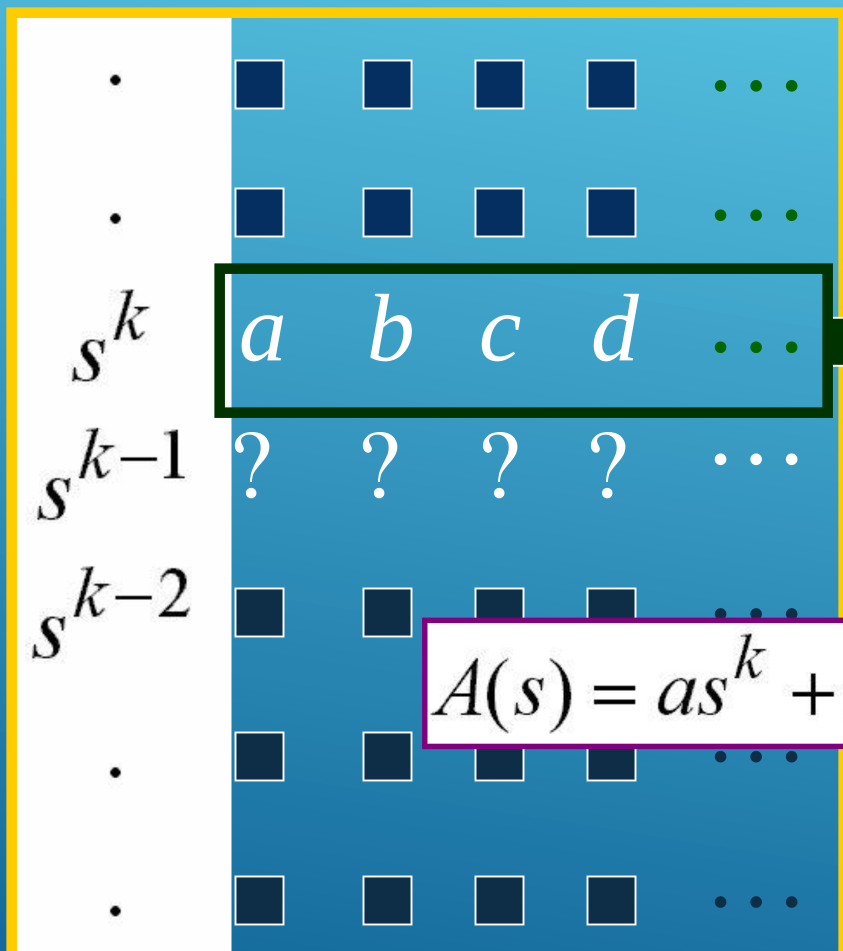
Handling the special cases
A row is entirely zero



Determine the auxiliary polynomial A(s) from this row

$$A(s) = as^k + bs^{k-2} + cs^{k-4} + ds^{k-6} + \cdots$$

Handling the special cases
A row is entirely zero

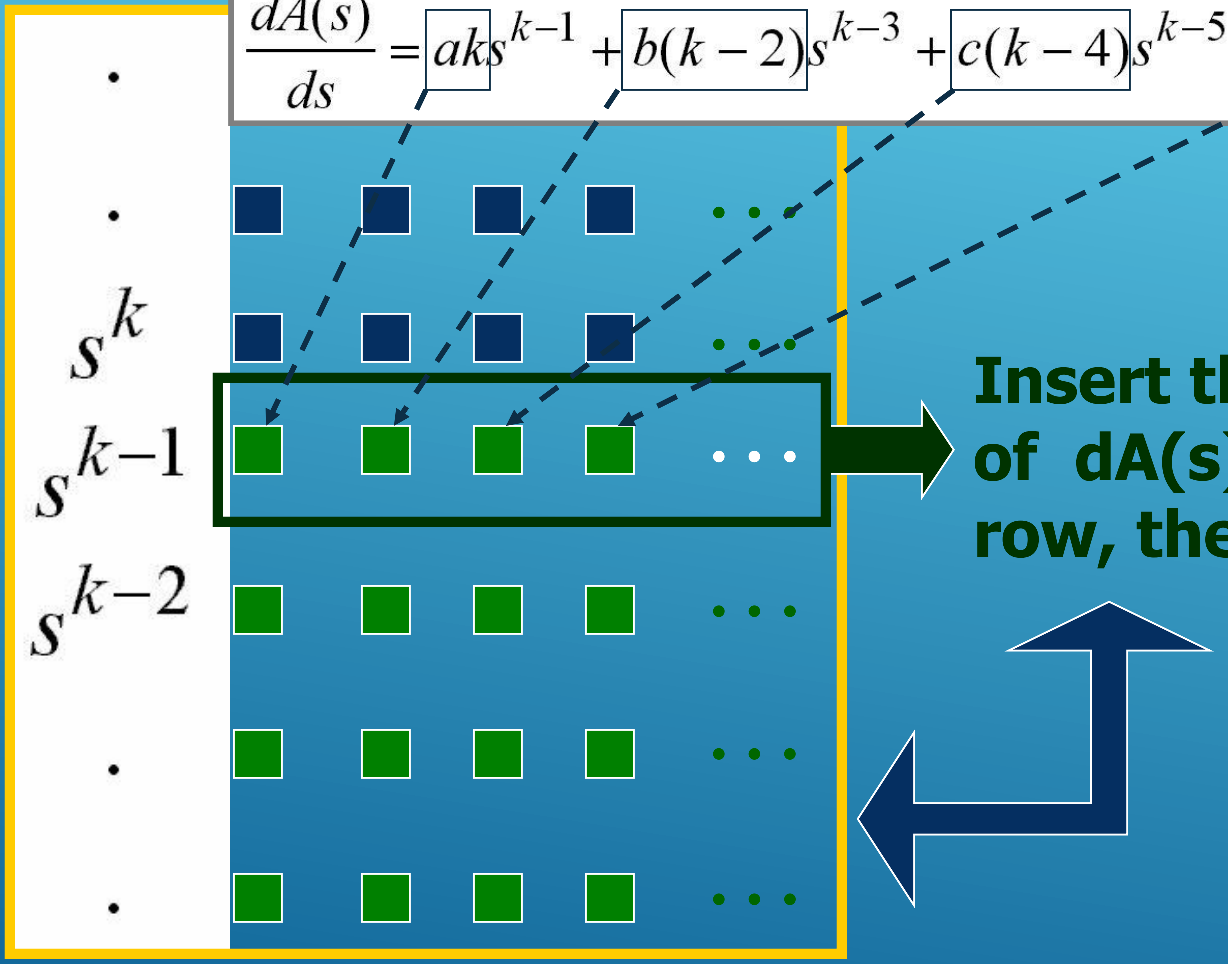


Determine the auxiliary polynomial A(s) from this row

$$A(s) = as^k + bs^{k-2} + cs^{k-4} + ds^{k-6} + \cdots$$

Handling the special cases
A row is entirely zero

$$\frac{dA(s)}{ds} = aks^{k-1} + b(k-2)s^{k-3} + c(k-4)s^{k-5} + d(k-6)s^{k-7} + \cdots$$



Insert the coefficients of dA(s)/ds into this row, then continue...

Handling the special cases - An Example
A row is entirely zero

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

$$A(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dA(s)}{ds} = 8s^3 + 96s$$

Handling the special cases - An Example
A row is entirely zero

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.6666	0	
s^0	-50		

$$\frac{dA(s)}{ds} = 8s^3 + 96s$$

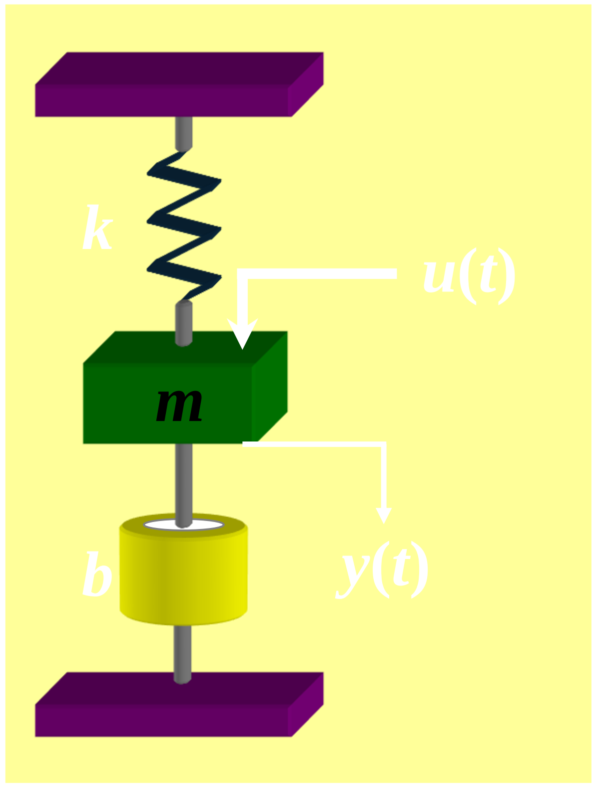
One sign change: One of the roots is in the right half s-plane

Final Remarks on Routh Criterion

- The goal of using Routh stability criterion is to explain whether the characteristic equation has roots on the right half s-plane.

- A parameter (e.g. a gain) may change the locations of the CL poles, and Routh criterion lets us know for which range the CL system is stable.

P-3 State Space Representation and Stability



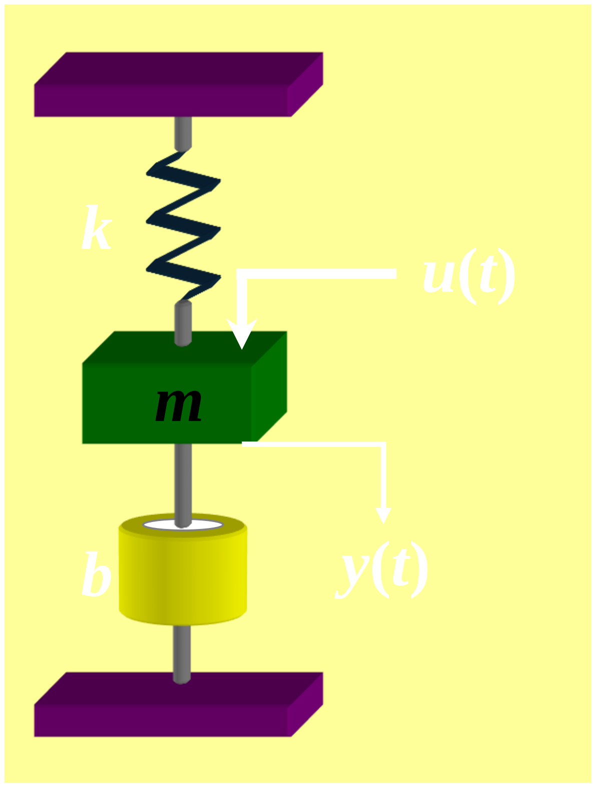
Consider the mass-spring-damper system. Laws of physics lead us to

$$m\ddot{y} + b\dot{y} + ky = u$$

Let us define the state as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

State Space Representation



Dynamics

$$m\ddot{y} + b\dot{y} + ky = u$$

State

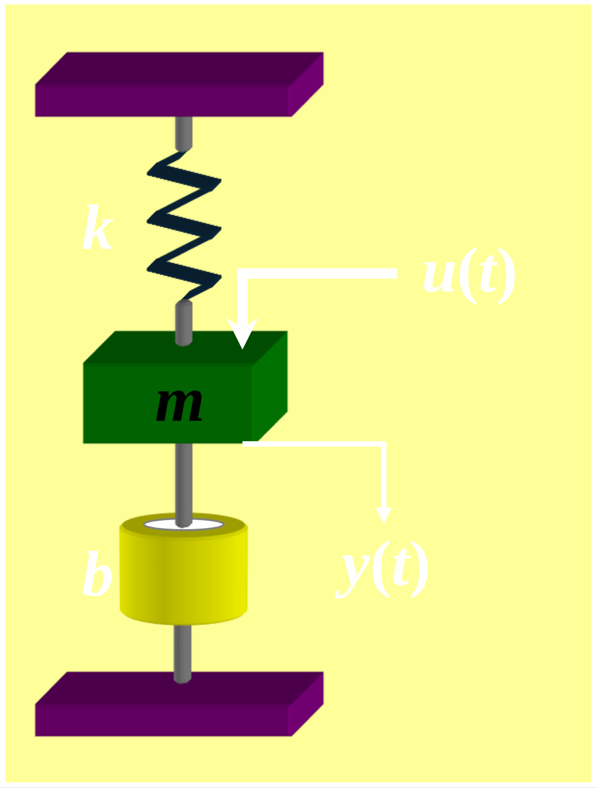
$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

State equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

Output equation $y = x_1$

State Space Representation



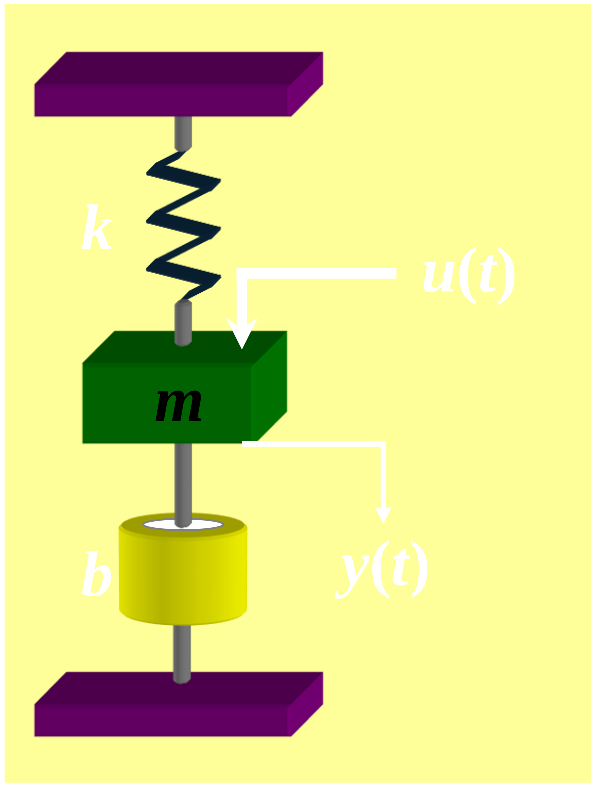
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Correlation between State Space Representations and Transfer Functions



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$(sI - A)X(s) = \cancel{x(0)} + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Correlation between State Space Representations and Transfer Functions

$$X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + 0$$

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

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Time Domain Dynamics

$$m\ddot{y} + b\dot{y} + ky = u$$

Relation between State Space Representations and Transfer Functions

What does this tell us?

Transfer Function

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

Time Domain Dynamics

$$m\ddot{y} + b\dot{y} + ky = u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

State Space Representation

Relation between State Space Representations and Transfer Functions

- The dynamics of a linear system can be expressed in any of the forms
  - Differential equations
  - Transfer functions
  - State space representation

One has to note that given the TF for a system, state space representation is not unique. Different realizations can be performed.

State Space Representation

State: The essence of past that influences the future. State is the smallest set of variables to describe the dynamics of a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

State Variables
The dimension of the state vector is fixed for a given system

State Space Representation

The dynamics of the system can uniquely be determined with the knowledge of $x_1(t_0)$, $x_2(t_0)$ and $u(t)$ for $t \geq t_0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

The state space is a space whose axes are the states. For the above example, axes are x_1 axis and x_2 axis.

State Space Representation

In general we have a set of differential equations

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$
$$\underline{y} = \underline{g}(\underline{x}, \underline{u}, t)$$

We linearize them and get

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$$
$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t)$$

The elements of the matrices may be time-varying

State Space Representation

We simply dropped the underlines. Clearly the state will be a vector if its dimension is larger than one.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad \qquad \dot{x} = Ax + Bu$$

$$\underline{y} = C\underline{x} + D\underline{u} \qquad \text{or} \qquad y = Cx + Du$$

Or may be time invariant

State Space Representation and Stability

Assume you are given the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- **The stability of this system can be determined by checking the eigenvalues of the matrix A**
- **Those eigenvalues are the poles of the transfer function**

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

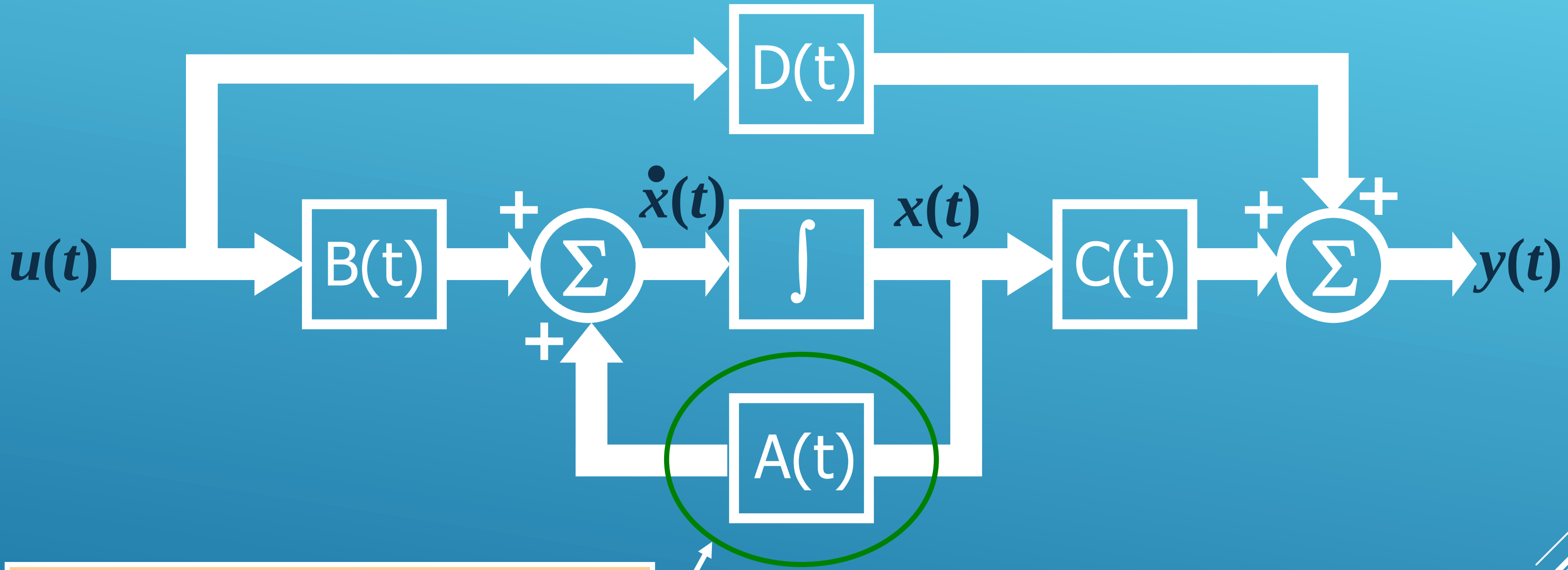
State Space Representation and Stability

$$\mathrm{eig}\{A\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$$

$$|\lambda I - A| = 0$$

- If $\mathrm{Re}\{\lambda_i\}<0$ for $i=1,2,\ldots,n$
  Then the system is stable
- If $\mathrm{Re}\{\lambda_i\}>0$ for some i
  Then the system is unstable
- If $\mathrm{Re}\{\lambda_i\}=0$ for some i
  Then the system has poles on the imaginary axis

State Space Representation and Stability
In summary...



Check the real parts of the eigenvalues of A(t)

An Example on Stability

$$\dot{x}_1 = x_2 - x_3$$
$$\dot{x}_2 = -x_1 + x_2 + x_3$$
$$\dot{x}_3 = ax_1 + x_2 - x_3 + u$$

$$y = x_1$$
$$T(s) = \frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)}$$

Determine the range of a for stability

An Example on Stability

$$\dot{x}_1 = x_2 - x_3$$
$$\dot{x}_2 = -x_1 + x_2 + x_3$$
$$\dot{x}_3 = ax_1 + x_2 - x_3 + u$$

$$y = x_1$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)}$$

$$sX_1(s) = X_2(s) - X_3(s)$$
$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$
$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

An Example on Stability

$$sX_1(s) = X_2(s) - X_3(s)$$

$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$

$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

$$X_3(s) = X_2(s) - sX_1(s)$$

$$X_2(s) = -\frac{s+1}{s-2}X_1(s)$$

An Example on Stability

$$X_3(s) = X_2(s) - sX_1(s)$$

$$X_2(s) = -\frac{s+1}{s-2}X_1(s)$$

$$X_3(s) = -\frac{s^2-s+1}{s-2}X_1(s)$$

An Example on Stability

$$sX_1(s) = X_2(s) - X_3(s)$$

$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$

$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

$$X_2(s) = -\frac{s+1}{s-2}X_1(s)$$

$$X_3(s) = -\frac{s^2 - s + 1}{s-2}X_1(s)$$

An Example on Stability

$$T(s) = \frac{2 - s}{s^3 + (a-1)s - 2a}$$

s^3	1	$a-1$
s^2	0	$-2a$
s^1		
s^0		

[redacted]

An Example on Stability

$$T(s) = \frac{2 - s}{s^3 + (a - 1)s - 2a}$$

s^3	1	$a - 1$
s^2	ε	$-2a$
s^1	$[\varepsilon(a-1) + 2a]/\varepsilon$	0
s^0	$-2a$	

An Example on Stability

$$T(s) = \frac{2 - s}{s^3 + (a-1)s - 2a}$$

s^3	1	$a-1$
s^2	ε	$-2a$
s^1	$a(1+2/\varepsilon)-1$	0
s^0	$-2a$	

This term becomes negative

$a < 0$

An Example on Stability

$$T(s) = \frac{2 - s}{s^3 + (a-1)s - 2a}$$

s^3	1	$a-1$
s^2	ε	$-2a$
s^1	$a(1+2/\varepsilon)-1$	0
s^0	$-2a$	

$a > 1/[1+2/\varepsilon]$

$\varepsilon > 0$ and $\varepsilon \approx 0$

$a > 0$

This term becomes negative

An Example on Stability

$$T(s) = \frac{2-s}{s^3 + (a-1)s - 2a}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ a & 1 & -1 \end{bmatrix}$$

The system is unstable regardless of the value of a. In other words, A has at least one eigenvalue in the right half s-plane

Can this system have poles on the imaginary axis?

$$T(s)=\frac{2-s}{s^3+(a-1)s-2a}$$

Assume the answer is yes… Then for $s=j\alpha$ the denominator must be zero, i.e.

$$(j\alpha)^3+(a-1)(j\alpha)-2a=0$$

$$j\left(-\alpha^3+(a-1)\alpha\right)-2a=0$$

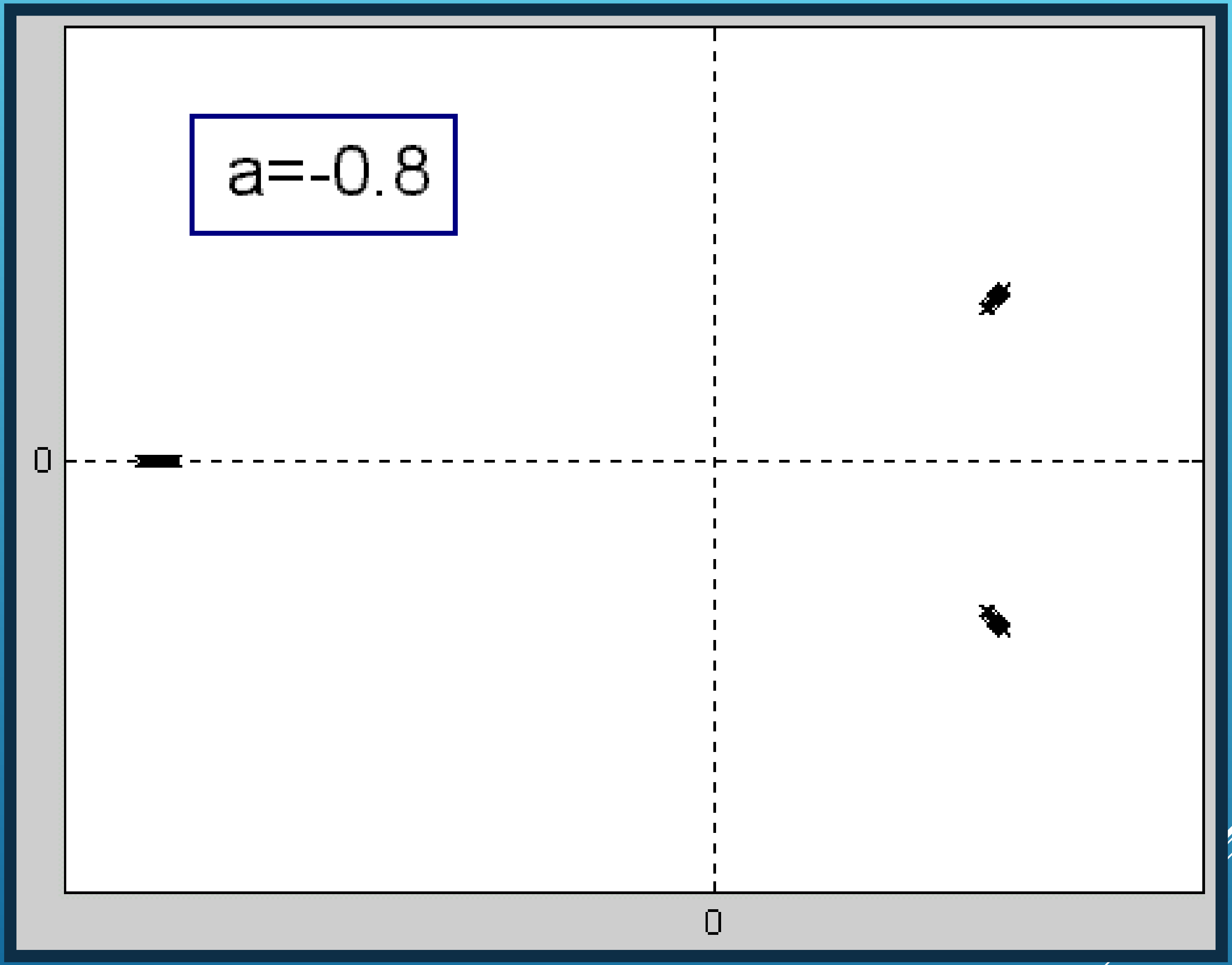
No value of a can lead to zero real and imaginary parts simultaneously

Can this system have complex conjugate poles on the imaginary axis?

$$T(s) = \frac{2 - s}{s^3 + (a-1)s - 2a}$$

The answer is no. Only one pole passes through the origin when a=0.

Watch now...



IMAGINARY AXIS

REAL AXIS