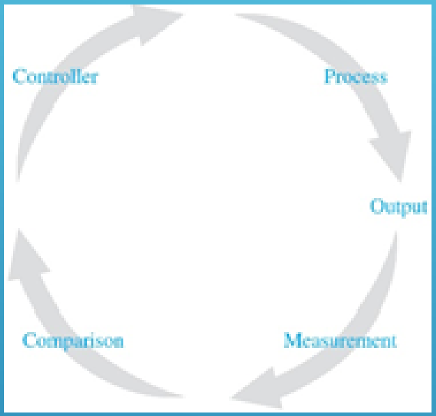
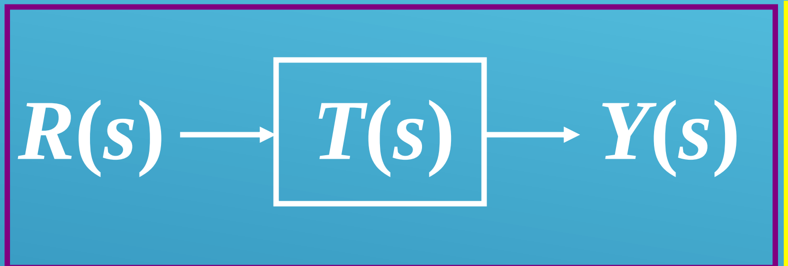
CONTROL SYSTEMS



Doç. Dr. Murat Efe

WEEK 8

Transient Response Analysis
Second Order Systems



$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **We will study**
  - **The unit step response, R(s)=1/s**
  - **The unit ramp response, R(s)=1/s^2**
  - **The unit impulse response, R(s)=1**
- **Clearly, Y(s)=T(s)R(s)**

Transient Response Analysis
Second Order Systems

Note that

$$T(s) = \frac{K}{Js^2 + Bs + K}$$

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$T(s) = \frac{K/J}{s^2 + (B/J)s + K/J}$$

$$\omega_n = \sqrt{K/J}$$

$$\zeta = (B/J)/\sqrt{4K/J}$$

Transient Response Analysis
Second Order Systems

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2$$

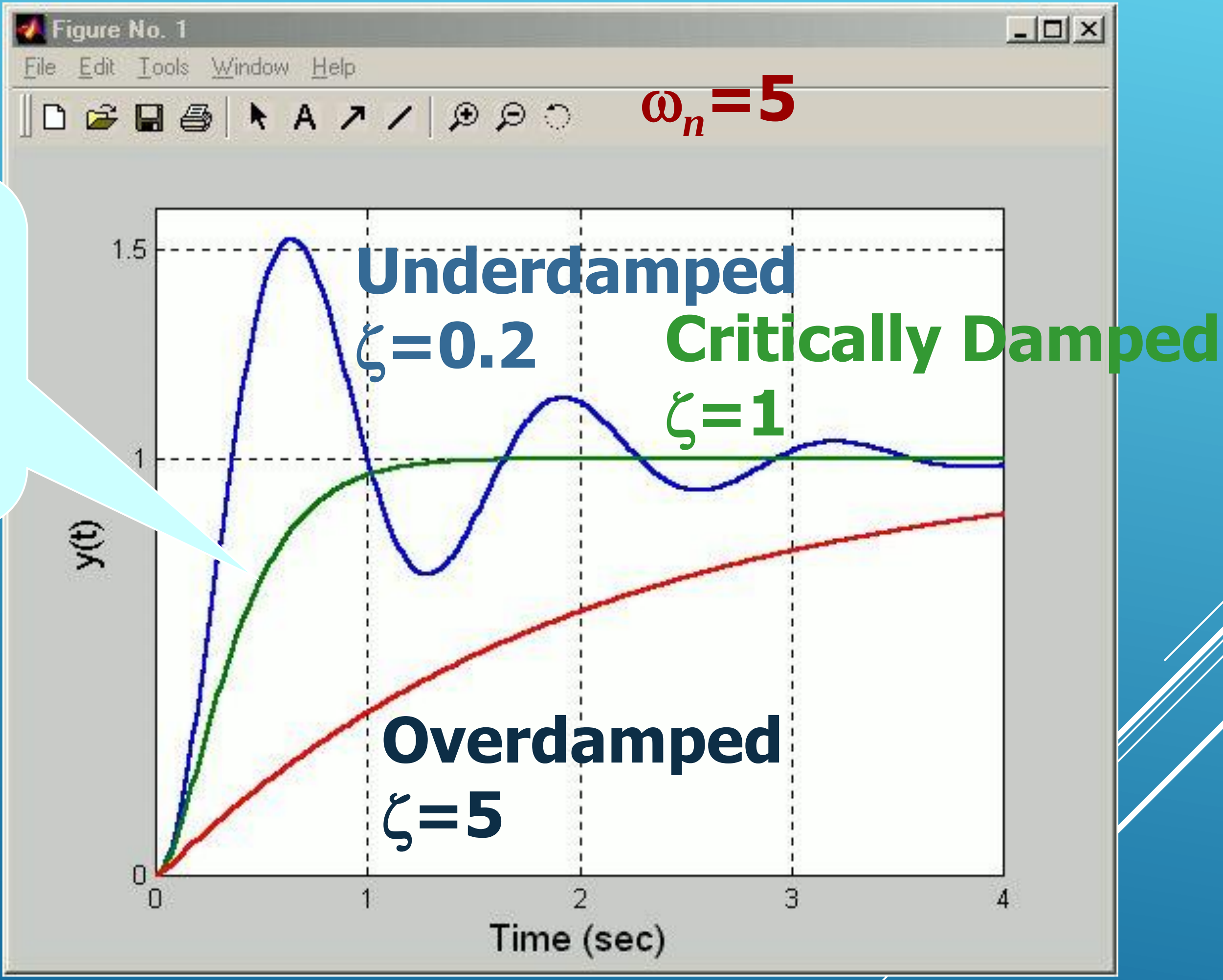
$$= 4\omega_n^2\left(\zeta^2 - 1\right)$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- $0 < \zeta < 1$ → Underdamped
- $\zeta = 1$ → Critically Damped
- $\zeta > 1$ → Overdamped

Transient Response Analysis
Second Order Systems, R(s)=1/s

Suspension system in a car needs to be critically damped



Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Damping ratio (ζ)

Natural frequency (ω_n)

Damped natural frequency (ω_d)

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s} = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

$$L^{-1}\left\{\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\} = e^{-\zeta\omega_n t}\cos(\omega_d t)$$

$$L^{-1}\left\{\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right\} = e^{-\zeta\omega_n t}\sin(\omega_d t)$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for } t \geq 0$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - 👓 Digression

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)$$

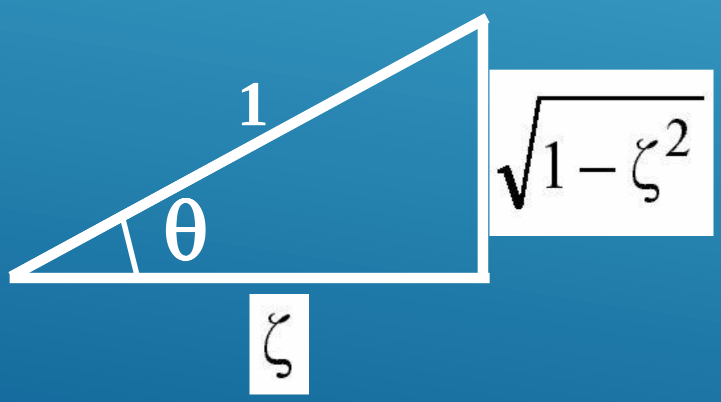
$\sqrt{1-\zeta^2}$ → **sin(θ)**

ζ → **cos(θ)**

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - Digression

$$\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t) = \sin(\omega_d t + \theta)$$

$$\sin(\theta) = \sqrt{1-\zeta^2} \text{ and } \cos(\theta) = \zeta$$



$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - Digression

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for } t \geq 0$$

- End of digression

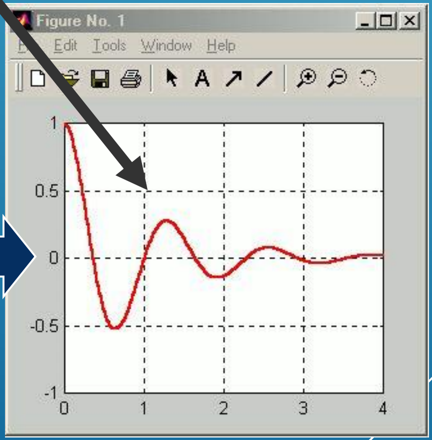
Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$e(t) = r(t) - y(t)$$

$$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \quad \text{for } t \geq 0$$

Oscillation frequency is ω_d

Damped sinusoidal oscillation converges to zero, $e(t) \to 0$



$\omega_n=5$
$\zeta=0.2$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Extreme Case (ζ=0, Undamped)

$$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \text{ for } t \geq 0$$

$$e(t) = \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \text{ for } t \geq 0$$

Oscillations continue indefinitely

Transient Response Analysis
Second Order Systems, R(s)=1/s
Critically Damped Case (ζ=1)

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s} = \frac{1}{s} - \frac{s+2\omega_n}{(s+\omega_n)^2}$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \quad \text{for } t \geq 0$$

$$y(t) = 1 - e^{-\omega_n t}\left(1+\omega_n t\right) \quad \text{for } t \geq 0$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>1)

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Two distinct poles on the negative real axis

$$\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2$$

$$= 4\omega_n^2\left(\zeta^2 - 1\right)$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$Y(s) = \frac{\omega_n^2}{\left(s - s_1\right)\left(s - s_2\right)s}$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>1)

$$Y(s) = \frac{\omega_n^2}{(s - s_1)(s - s_2)s}$$

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

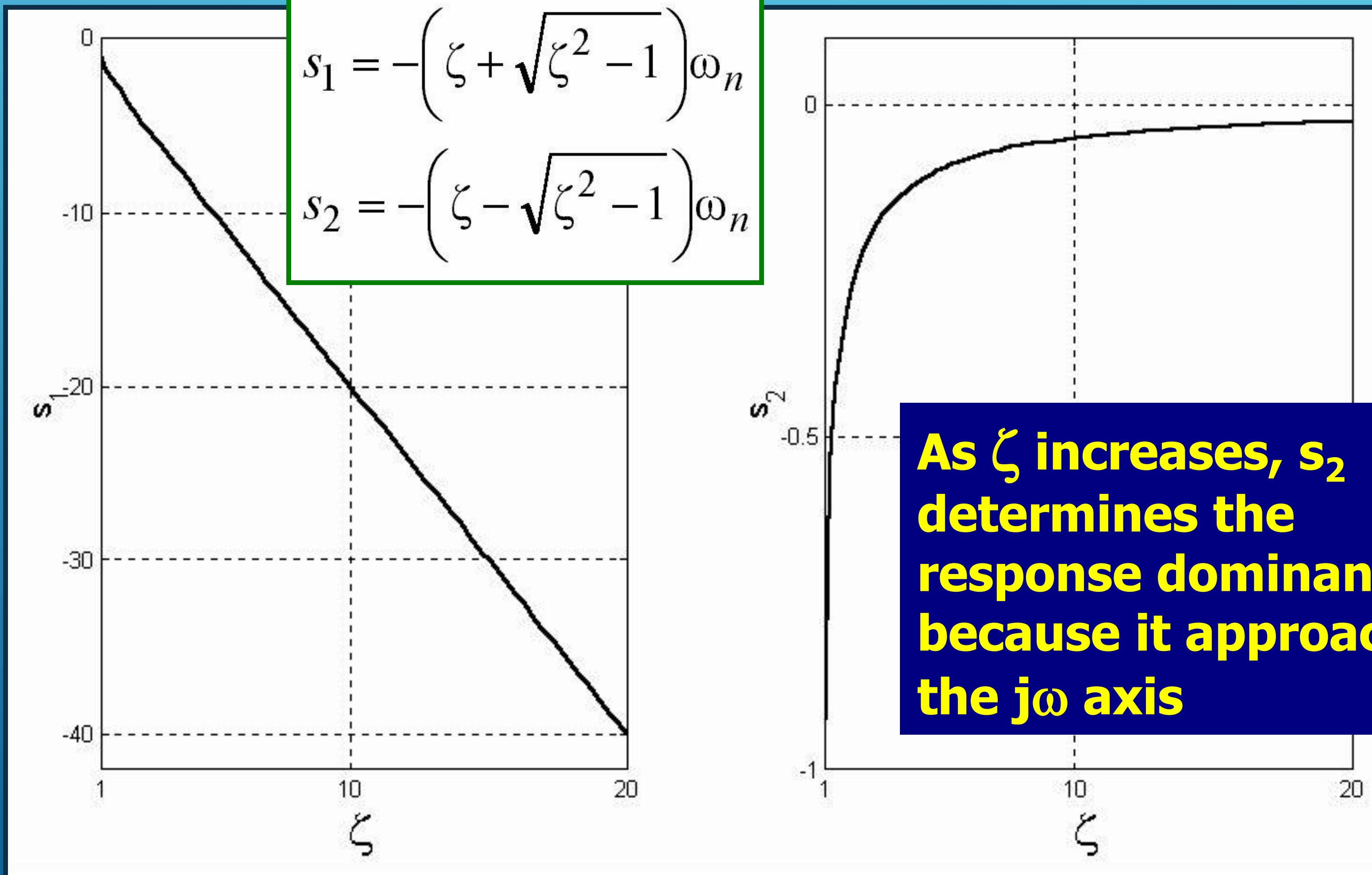
$$Y(s) = \frac{1}{s} + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{1/s_2}{s - s_2} - \frac{1/s_1}{s - s_1}\right)$$

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{s_2 t}}{s_2} - \frac{e^{s_1 t}}{s_1}\right)$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>1). See $s_{1,2}$ for ω_n=1

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$



As ζ increases, s_2 determines the response dominantly, because it approaches the $j\omega$ axis

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case ($\zeta>>1$)

$$T(s)=\frac{\left(\frac{s_1 s_2}{s_1-s_2}\right)}{s-s_1}+\frac{\left(\frac{s_1 s_2}{s_2-s_1}\right)}{s-s_2}$$

When $\zeta>>1$

$s_1 \to -2\zeta\omega_n$

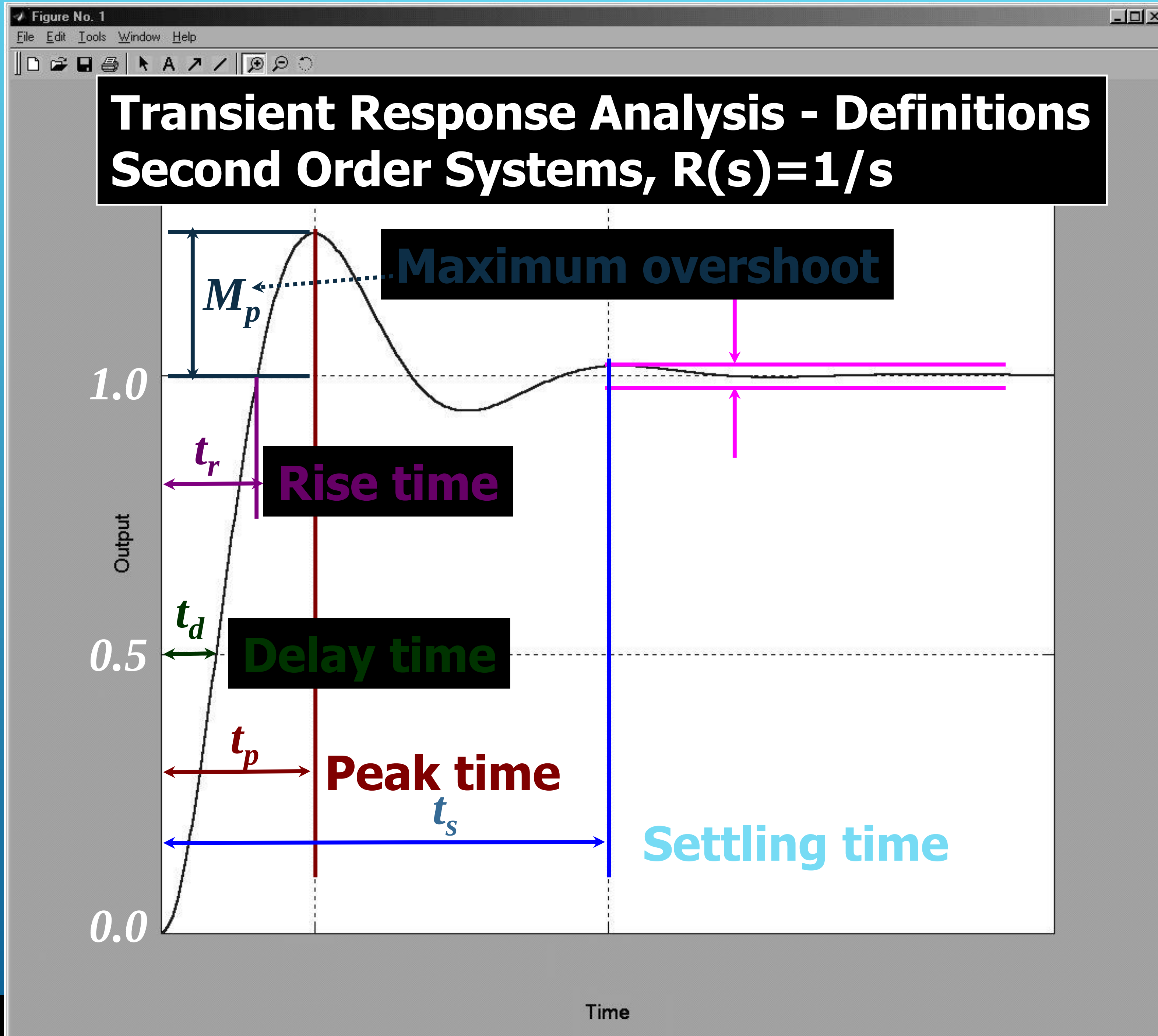
$s_2 \to 0$

$$T(s)\cong\frac{\left(\frac{s_2}{s_2/s_1-1}\right)}{s-s_2}\cong\frac{-s_2}{s-s_2}$$

$$y(t)=1-e^{s_2 t}$$

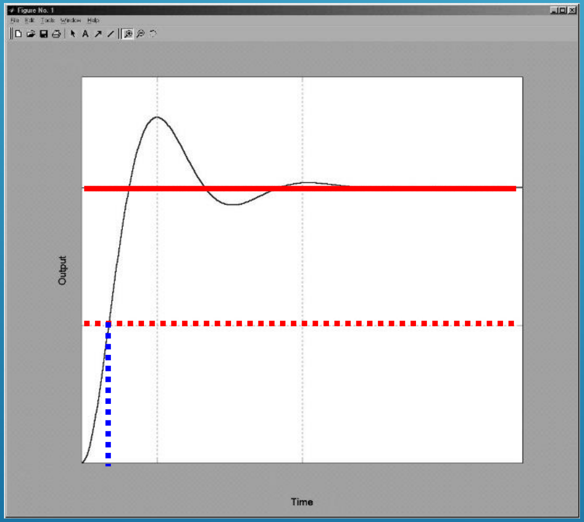
y(0)=0, y(∞)=1 are satisfied by an approximate dominant first order dynamics

Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s



Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Delay Time (t_d): The time required to reach the half of the final value. Note that delay time is the time till first reach is observed.



Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

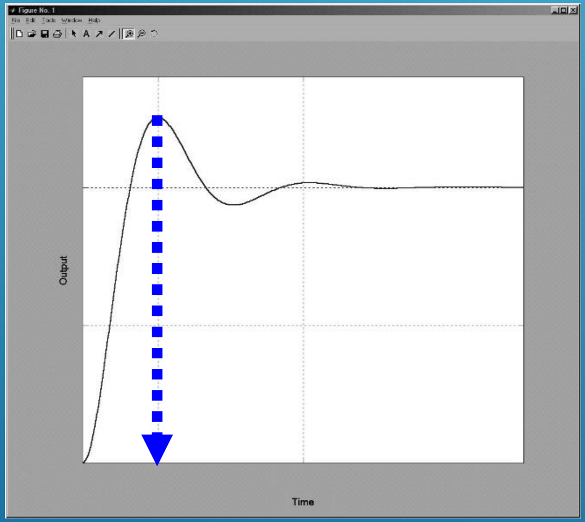
Rise Time (t_r): The time required to rise from 10% to 90% or 5% to 95% or 0% to 100% of the final value.

Generally for underdamped 2nd order systems

Generally for overdamped systems

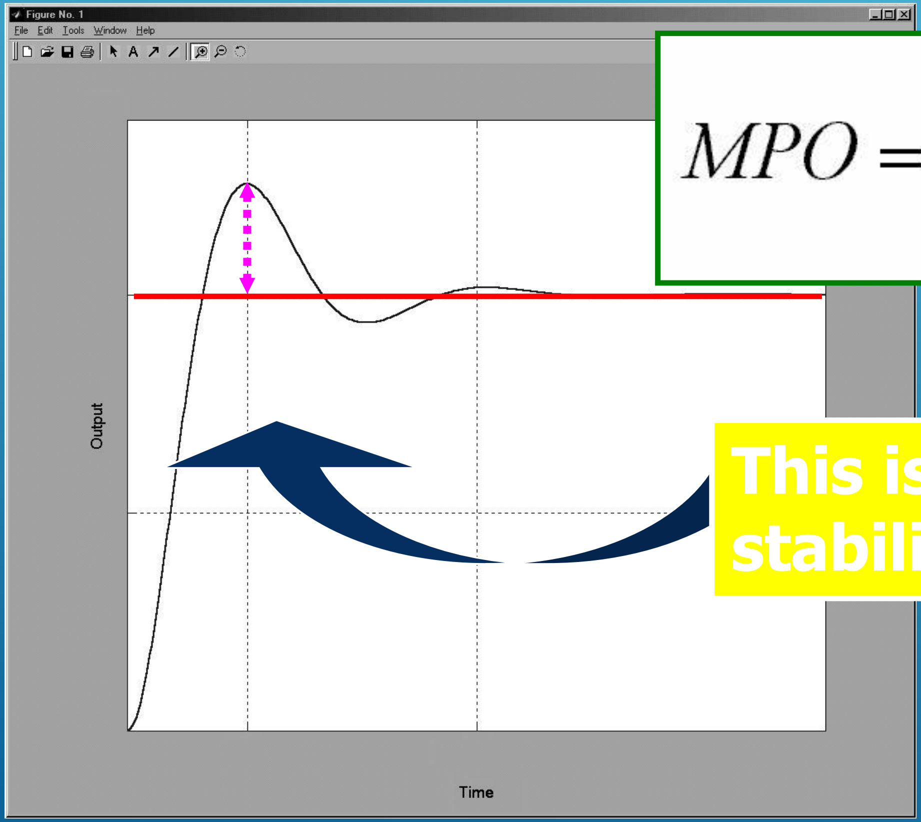
Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Peak Time (t_p): The time required for the response to reach the first peak of the overshoot.



Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Maximum (percent) Overshoot (M_p): The maximum peak value measured from the steady state value.

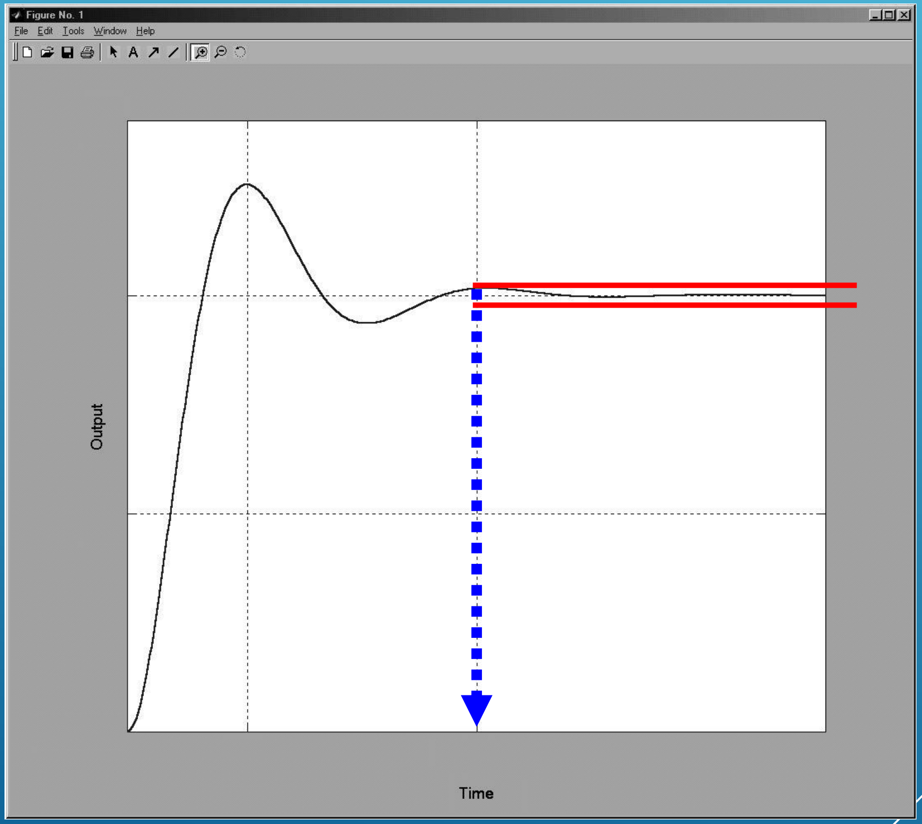


$$MPO = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

This is a measure of relative stability of the system

Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Settling Time (t_s): The time required for the response to remain within a desired percentage (2% or 5%) of the final value.



Transient Response Specifications
Second Order Systems, R(s)=1/s

In a control system, the designer may want to observe some set of <u>predefined</u> transient response characteristics. This section focuses on the computation of the variables of transient response and their relevance to <u>closed loop transfer function</u>. Ultimately, this relevance will bring a set of constraints for the design of the <u>controller</u>.

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Rise Time (t_r)

$$y(t_r) = 1 = 1 - e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right)$$

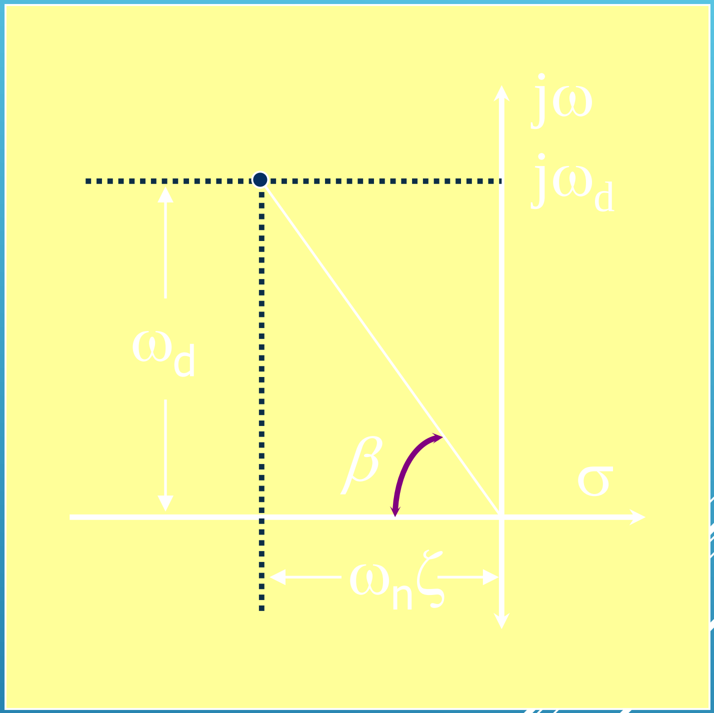
$$e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right) = 0$$

$$\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r) = 0 \Rightarrow \tan(\omega_d t_r) = -\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Rise Time (t_r)

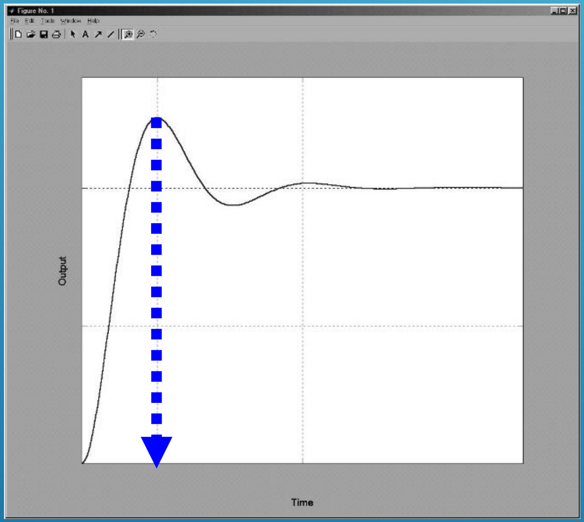
$$\tan(\omega_d t_r) = -\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta} = -\frac{\omega_d}{\omega_n\zeta}$$

$$t_r = \frac{1}{\omega_d}\arctan\left(-\frac{\omega_d}{\omega_n\zeta}\right) = \frac{\pi-\beta}{\omega_d}$$



Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Peak Time (t_p)

At $t=t_p$, $dy/dt=0$



$$\frac{dy(t)}{dt} = \zeta\omega_n e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) + e^{-\zeta\omega_n t}\left(\omega_d \sin(\omega_d t) - \frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos(\omega_d t)\right)$$

$$\frac{dy(t_p)}{dt} = \sin(\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Peak Time (t_p)

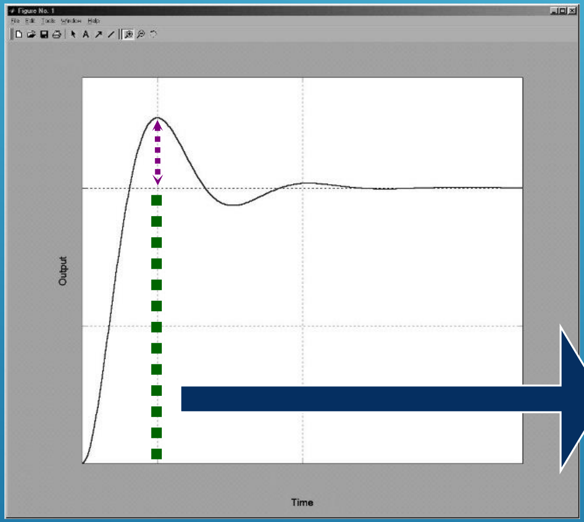
$$\frac{dy(t_p)}{dt} = \sin(\omega_d t_p) \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} = 0$$

$$\sin(\omega_d t_p) = 0 \quad \text{or} \quad \omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

$$t_p = \frac{\pi}{\omega_d}$$

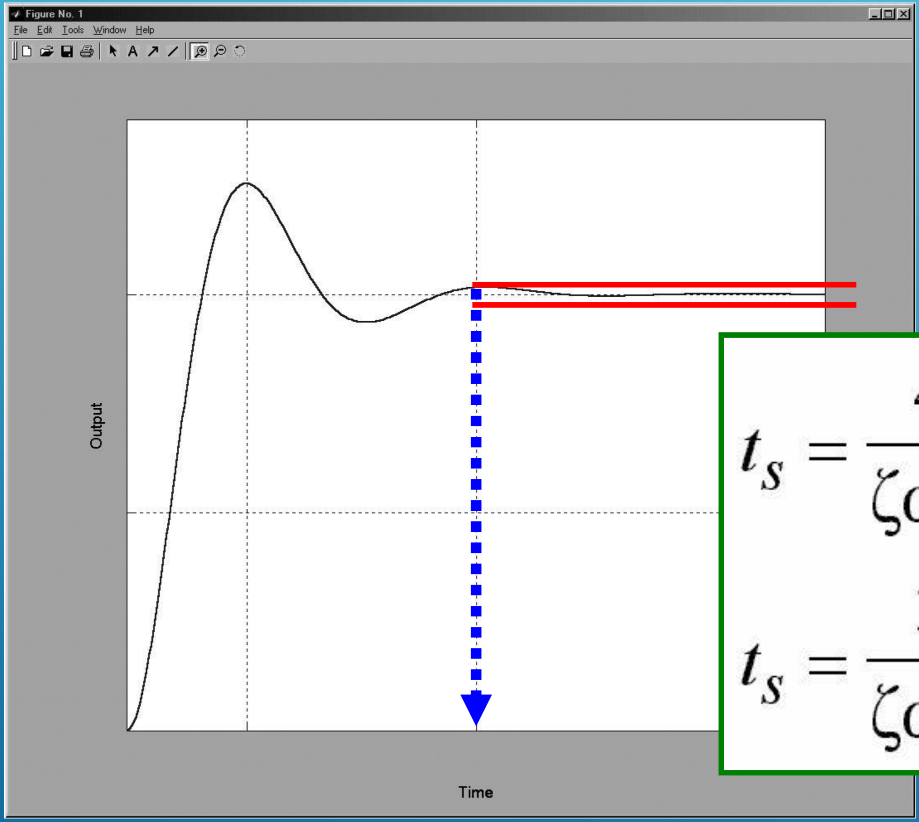
Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Maximum Overshoot (M_p)



$$M_p = y(t_p) - 1 = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

Note that maximum overshoot occurs at $t=t_p$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Settling Time (t_s)



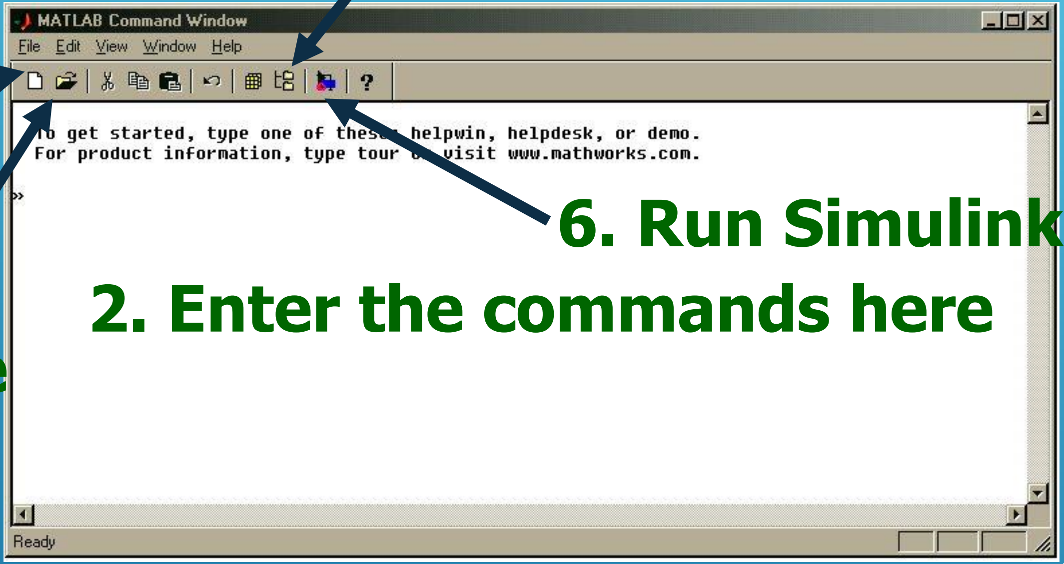
$$t_s = \frac{4}{\zeta\omega_n}$$ → **2% Criterion**

$$t_s = \frac{3}{\zeta\omega_n}$$ → **5% Criterion**

Using Matlab with Simulink

1. Set your path

3. Create a new m-file



6. Run Simulink

4. Open an existing m-file

2. Enter the commands here

Using Matlab with Simulink

Try these first, see the results

MATLAB Command Window

File Edit View Window Help

```
To get started, type one of these: helpwin, helpdesk, or demo.
For product information, type tour or visit www.mathworks.com.

>> A=[3 7;8 14]

A =

     3     7
     8    14

>>
```

»inv(A)

»det(A)

»eig(A)

»A.^2

»A^2

»A*A

»sum(A)

»sum(sum(A))

»A'

»A(:,1)

»A(2,:)

»diag(A)

»A(1,1)*A(1,2)

»A^3

»i*A

»A+eye(2,2)

Ready

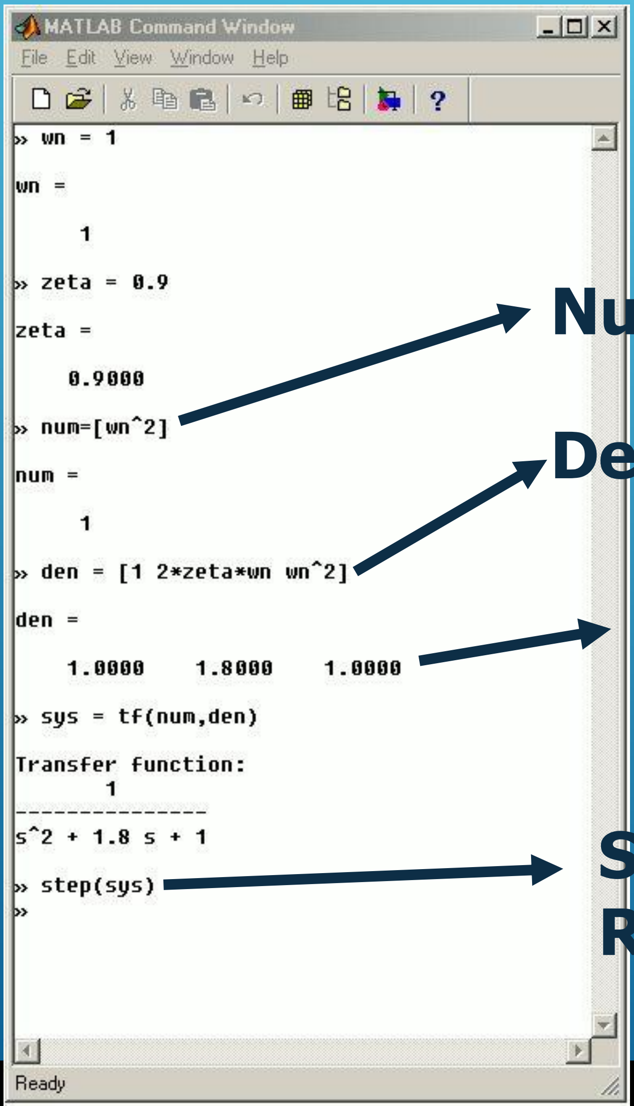
Using Matlab with Simulink
Useful commands/examples

- » clc
- » clear
- » figure
- » help {keyword}
- » close all
- » size(A)
- » rand(3,2)
- » real(a)
- » imag(a)
- » grid
- » zoom
- » clf

- » max(A)
- » min(A)
- » flops
- » who
- » whos
- » sin(pi/2)
- » cos(1.34)
- » atan(1.34)
- » abs(-2)
- » log(3)
- » log10(3)
- » sign(-2)

- » save
- » zeros(3,1)
- » ones(2,4)
- » ceil(1.34)
- » floor(1.34)
- » ezplot('sin(x)',[0,2])
- » helpdesk
- » roots([1 7 10])
- » ltiview
- » rlocus
- » nyquist
- » bode
- » margin

Using Matlab with Simulink
A command line demo - Step Response

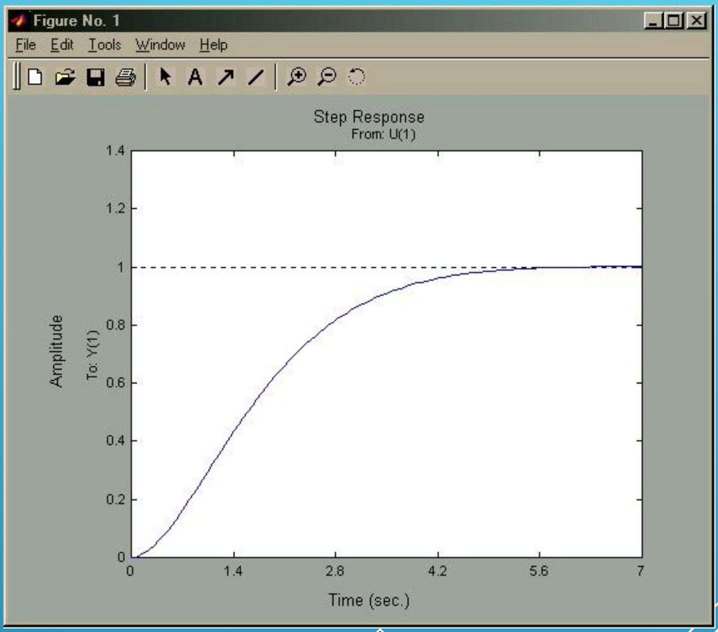


Numerator

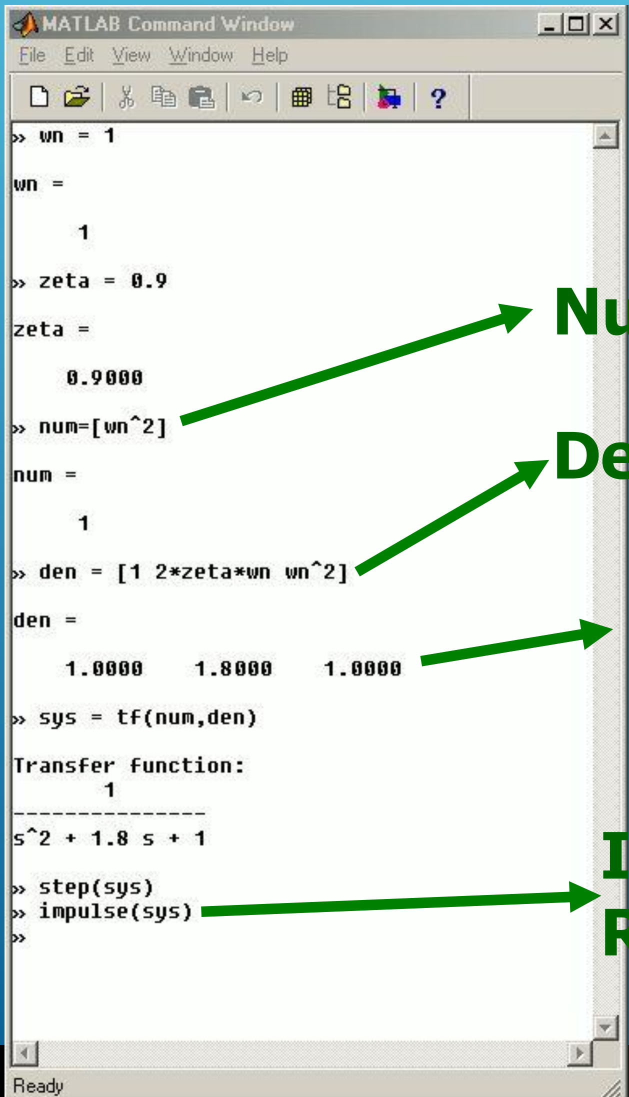
Denominator

Transfer Function

Step Response



Using Matlab with Simulink
A command line demo - Impulse Response

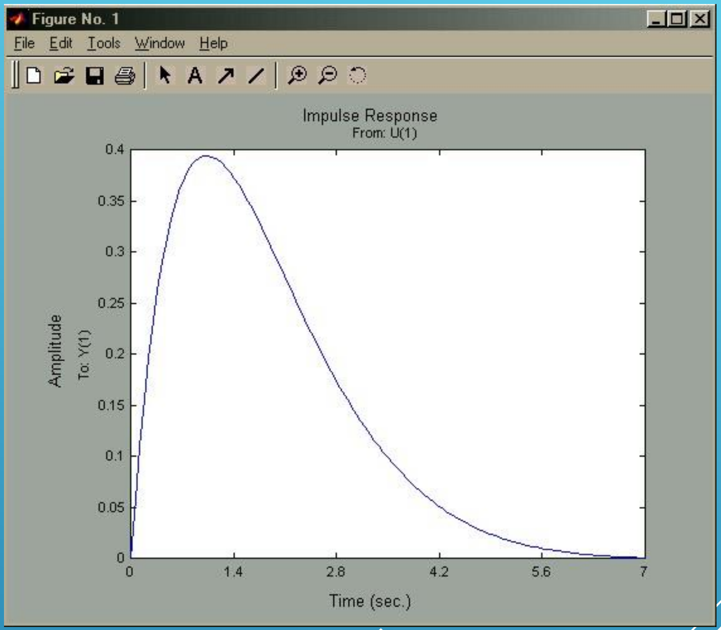


Numerator

Denominator

Transfer Function

Impulse Response



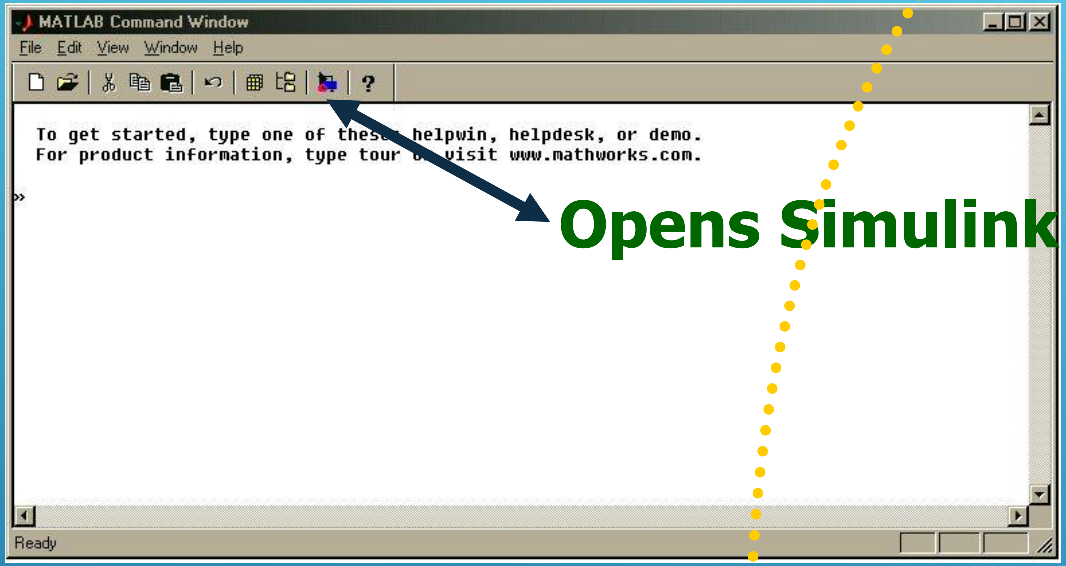
Using Matlab with Simulink

Type »help toolbox/control
To see all *control systems* related functions and library tools

Type »help elmat
To see *elementary matrix* operators and related tools

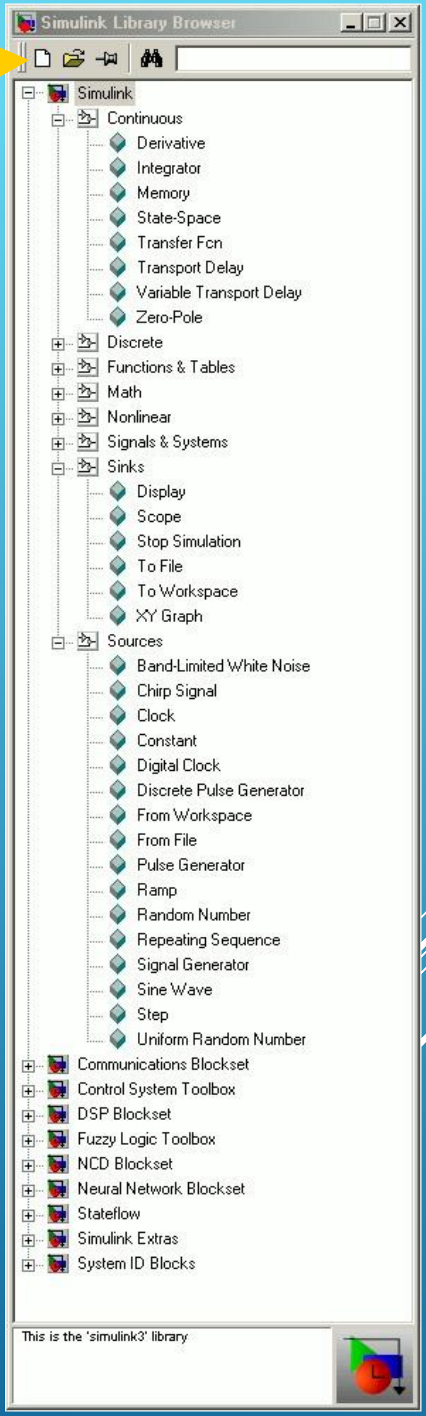
Using Matlab with Simulink

Simulink



Opens Simulink

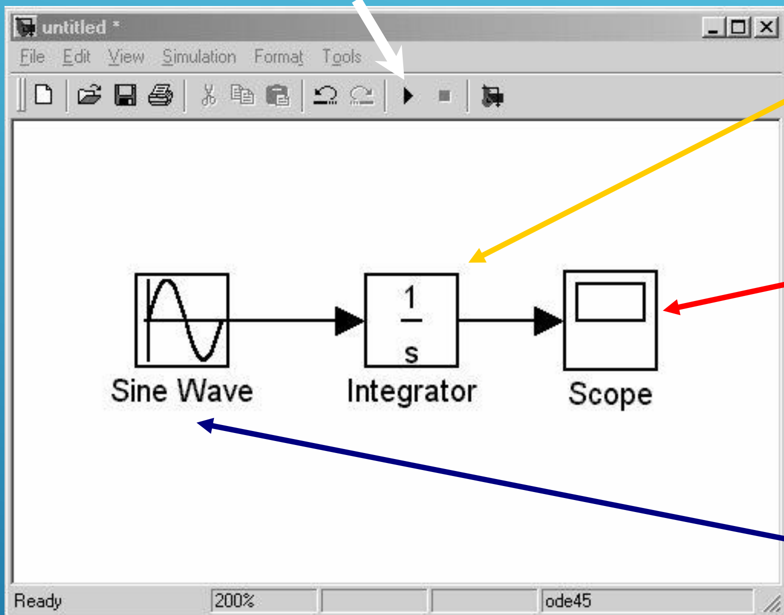
Creates a new model

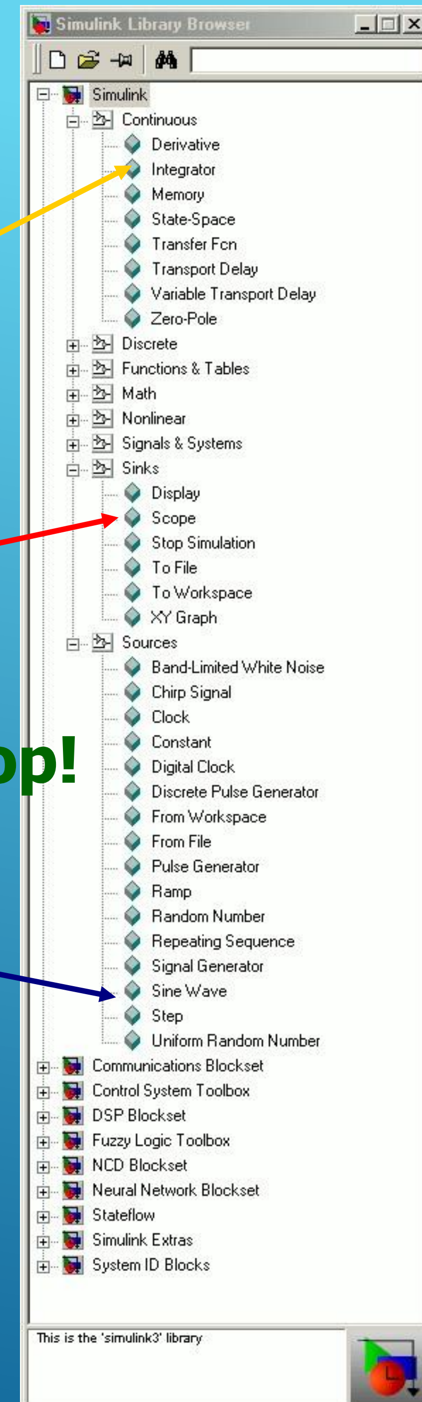


Using Matlab with Simulink

Simulink

4. Run the model





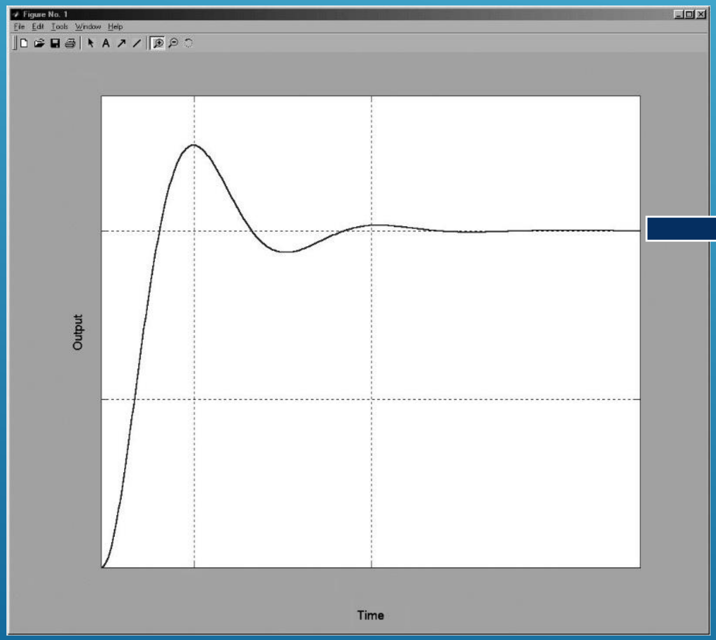
1. Drag & Drop!

2. Connect the components

3. Double click to set the internal parameters (e.g. magnitude or phase of sine wave, initial value of the integrator etc.)

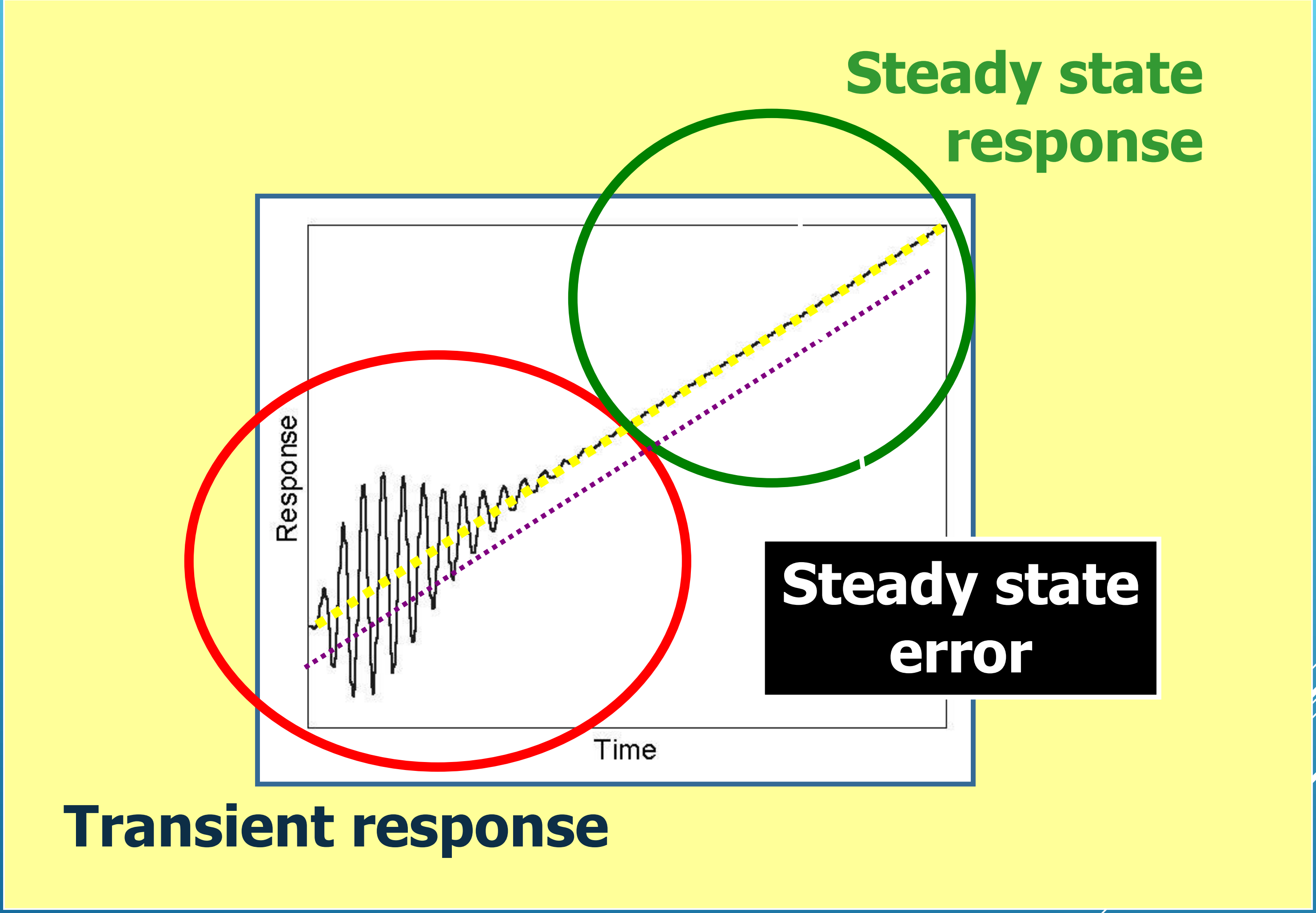
P-4 Steady State Errors

- **Steady state response is the manner in which the system output behaves as time approaches infinity**



This is the steady state value

Steady State Errors



Steady State Errors

- Control systems can be classified according to their ability to follow several test inputs.
- We will analyze the steady state error for certain types of inputs, such as step, ramp or parabolic commands.
- Most input signals can be written as combinations of these signals, so the classification is reasonable.

Steady State Errors

- **Whether a given control system will exhibit steady state error for a given type of input depends on the type of open loop transfer function of the system.**

- **Type of open loop transfer function is the number of integrators contained.**

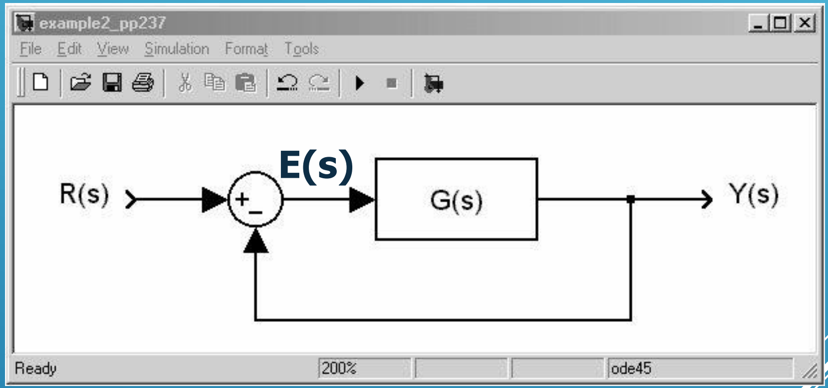
$$G(s) = \frac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{s^N(s+a_1)(s+a_2)\cdots(s+a_n)}$$

Steady State Errors

$$G(s)=\frac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{s^N(s+a_1)(s+a_2)\cdots(s+a_n)}$$

We will consider only

N=0 → Type 0
N=1 → Type 1
N=2 → Type 2

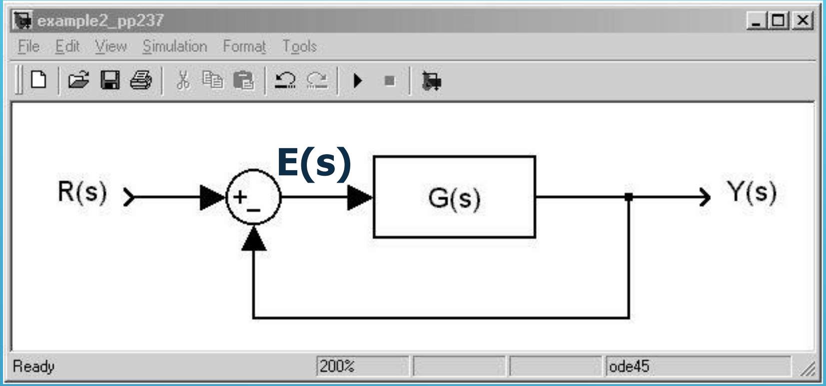


$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

$$E(s)=\frac{1}{1+G(s)}R(s)$$

Steady State Errors

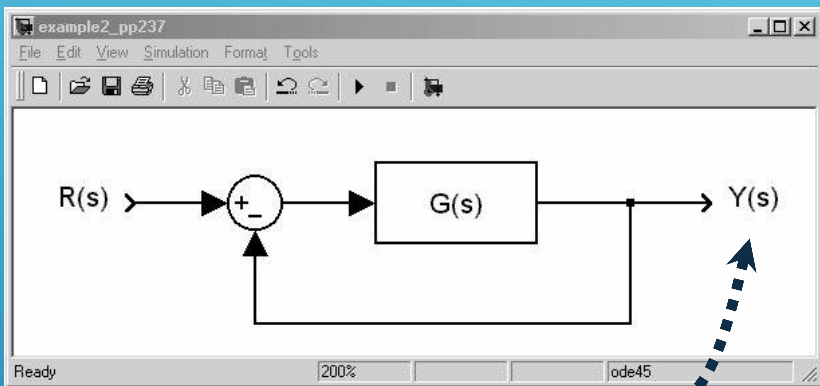
$$E(s) = \frac{1}{1+G(s)} R(s)$$



$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Final Value Theorem

Steady State Errors



Regardless of the corresponding physics, we will consider position, velocity and acceleration outputs

Position	$\frac{d}{dt}$	Velocity	$\frac{d}{dt}$	Acceleration
Pressure		Pressure Change		Change in Pressure Change
Temperature		Temperature Change		Change in Temperature Change

Steady State Errors
Static Position/Velocity/Acceleration
Error Constants

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s} \qquad K_p = \lim_{s\to 0} G(s) = G(0) \qquad e_{ss} = \frac{1}{1+K_p}$$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^2} \qquad K_v = \lim_{s\to 0} sG(s) \qquad e_{ss} = \frac{1}{K_v}$$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)} \frac{1}{s^3} \qquad K_a = \lim_{s\to 0} s^2G(s) \qquad e_{ss} = \frac{1}{K_a}$$

The larger the constants, the smaller the e_{ss}

Steady State Errors
Static Position/Velocity/Acceleration Error Constants

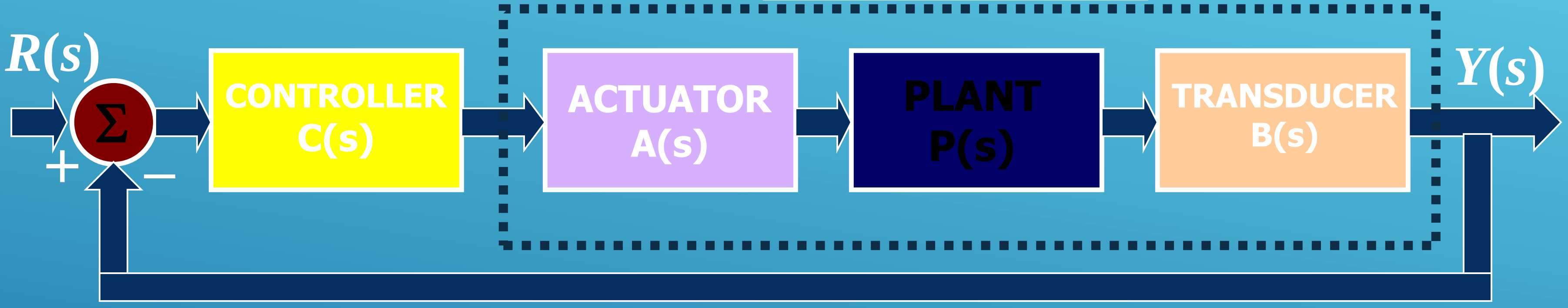
System Type \ Input Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\frac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\frac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\frac{1}{K_a}$

Transient Response
Steady State Response

We analyzed the characteristics of the response of the closed loop system. In any practical design, you will have a number of design specifications, which may impose penalties on transient or steady state characteristics.

An Example

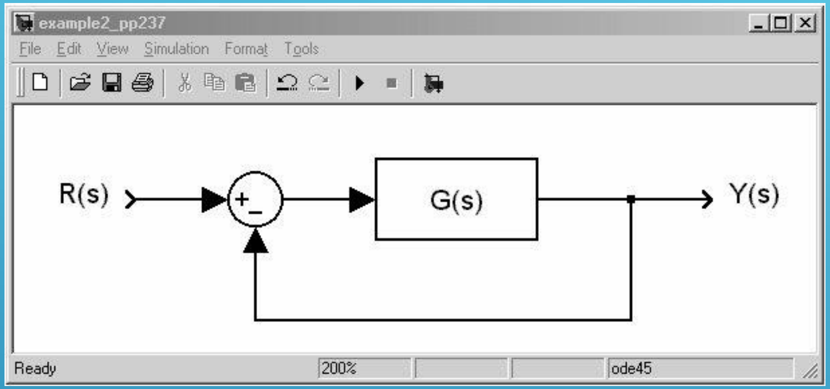
$$A(s) = K_1 \qquad P(s) = \frac{a}{s(s+b)} \qquad B(s) = \frac{K_2}{s+K_2}$$



$$G(s) = K_1 \frac{a}{s(s+b)} \frac{K_2}{s+K_2} C(s)$$

Open Loop Transfer Function

An Example



$$G(s) = \frac{K_1 K_2 a}{s(s+b)(s+K_2)} C(s)$$

$$K_1 = 10, K_2 = 20, a = 1, b = 4$$

Design a PD controller such that

- **The closed loop system becomes stable**
- **The closed loop system follows the unit ramp with minimum possible steady state error**
- **Response of the closed loop for unit step input exhibits maximum overshoot M_p=0.1**

These are the specifications of the design...

An Example
Stability Requirement

Choose controller as → $C(s) = K(s+20)$

Open Loop TF → $G(s) = \frac{200K}{s(s+4)}$

Closed Loop TF

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{200K}{s^2 + 4s + 200K}$$

s^2	1	$200K$
s^1	4	0
s^0	$200K$	

→ $K > 0$

An Example
Steady State Error Requirement
Obtain minimum e_{ss} for ramp input

$$E(s) = \frac{1}{1+\frac{200K}{s(s+4)}}\frac{1}{s^2} = \frac{s^2+4s}{s^2+4s+200K}\frac{1}{s^2}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s+4}{s^2+4s+200K} = \frac{1}{50K}$$

Should you choose K as large as possible?

An Example
Maximum Overshoot Requirement

Closed Loop TF

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{200K}{s^2 + 4s + 200K}$$

$$M_p = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.1$$

$$\zeta \cong 0.5912$$

$$2\zeta\omega_n = 4$$

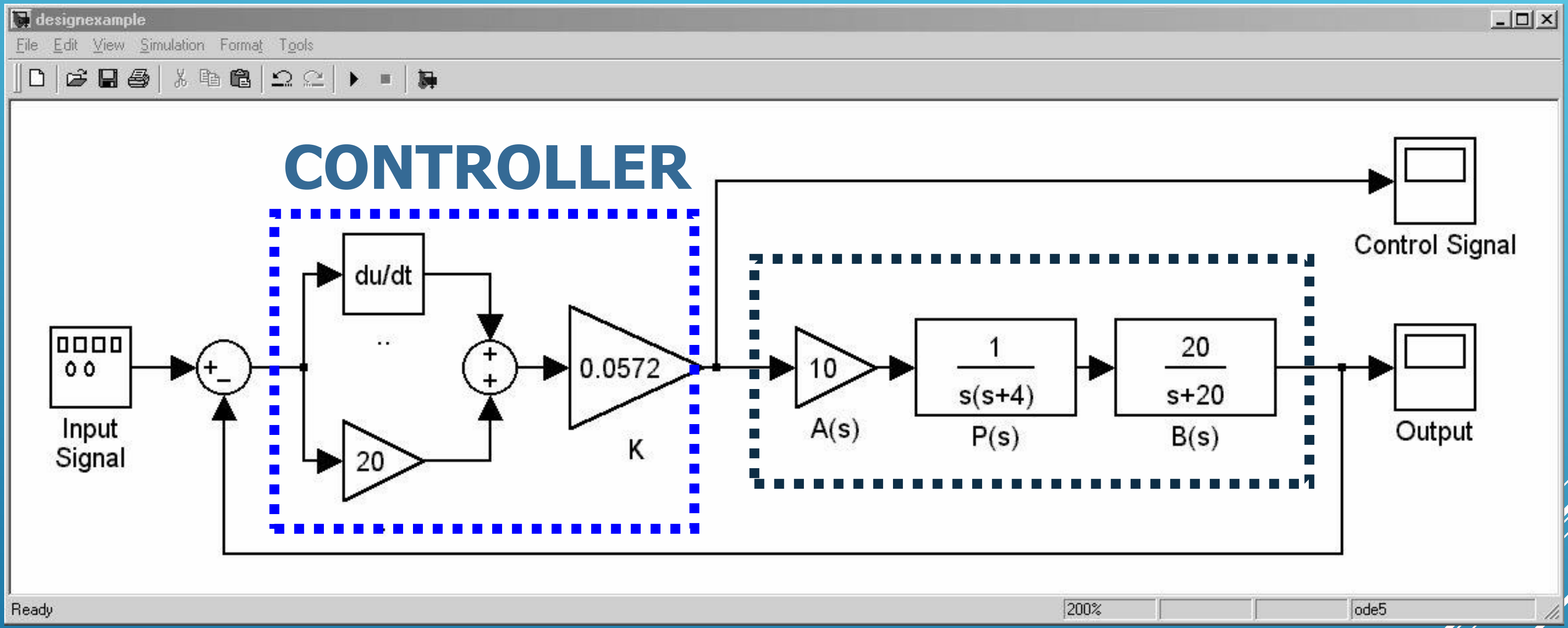
$$\omega_n^2 = 200K$$

$$\omega_n \cong 3.3832$$

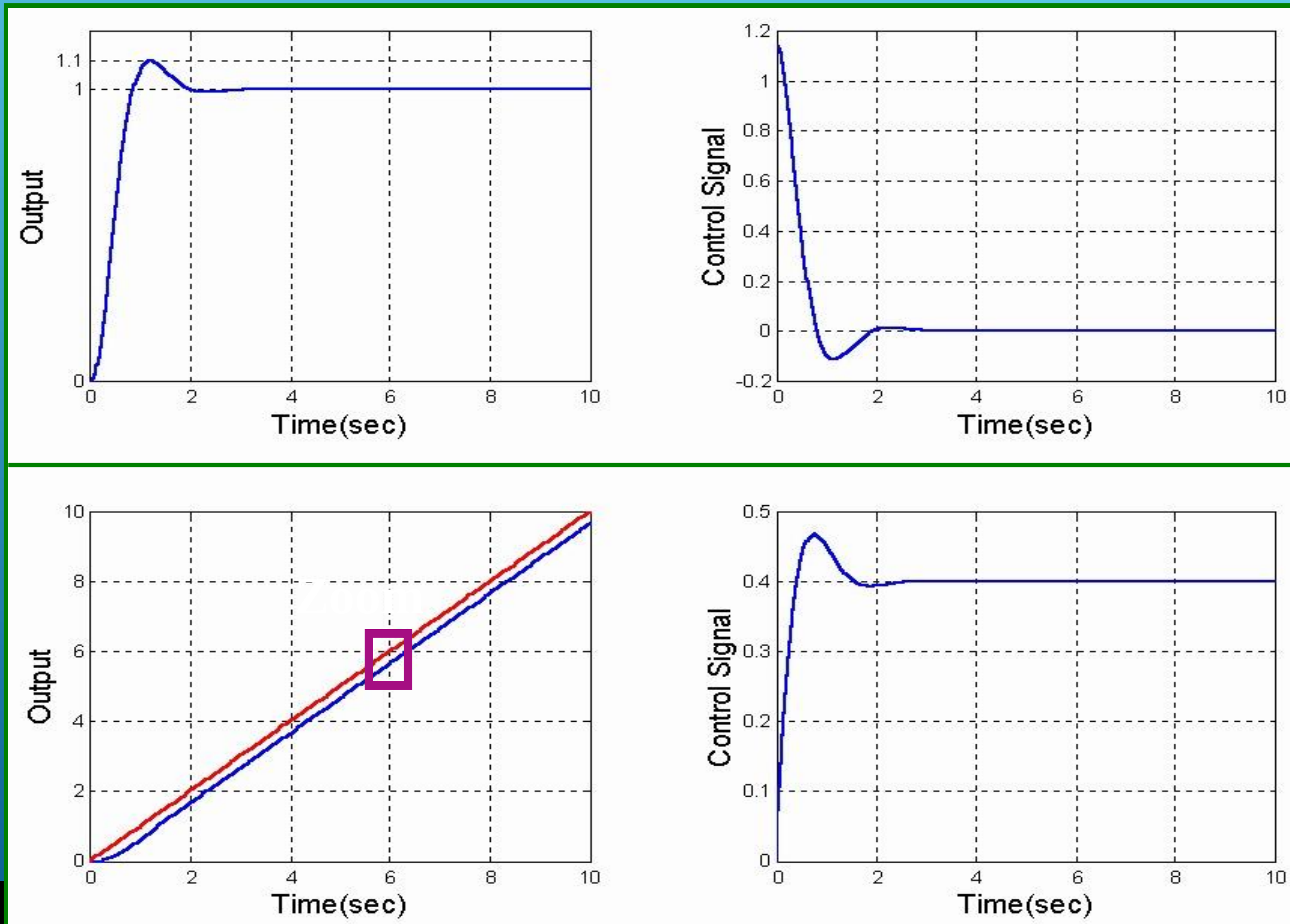
$$K \cong 0.0572$$

$$e_{ss} = \frac{1}{50K} \cong 0.3495$$

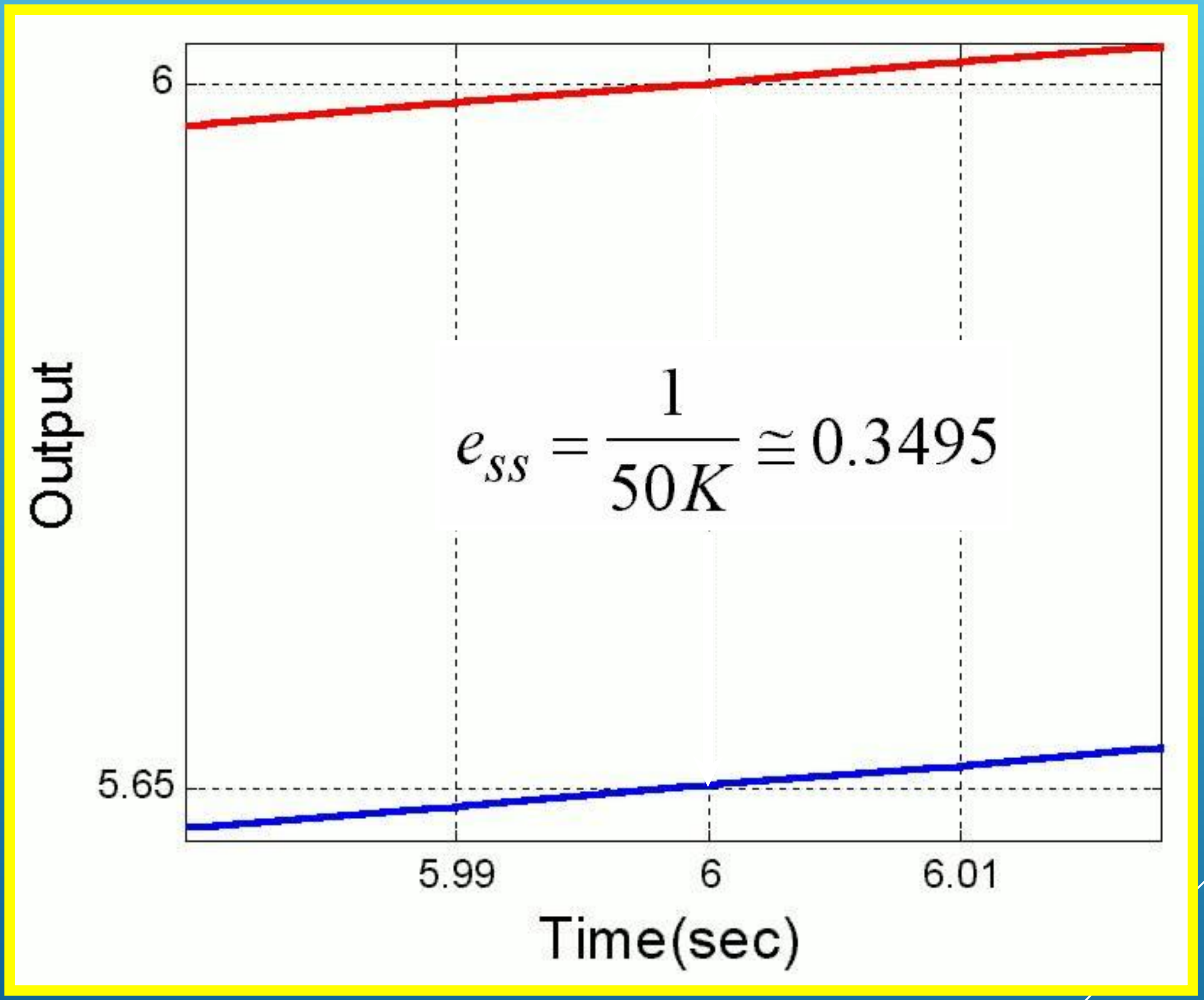
An Example
Justification of the Design



An Example
Justification of the Design



An Example
Justification of the Design



An Example
Remarks

Controller is → $C(s) = K(s+20)$

Open Loop TF is

$$G(s) = \frac{K_1 K_2 a}{s(s+b)(s+K_2)} C(s)$$

The product of them cancels out the pole at $s=-K_2$. Never cancel an unstable pole! Since $K_2>0$, we could do it. If K_2 were negative, an imperfect cancellation would result in instabilities in the long run; and in practice, we are always faced to imperfections!