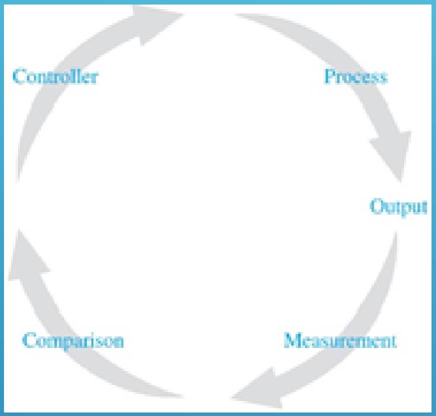
CONTROL SYSTEMS



Doç. Dr. Murat Efe

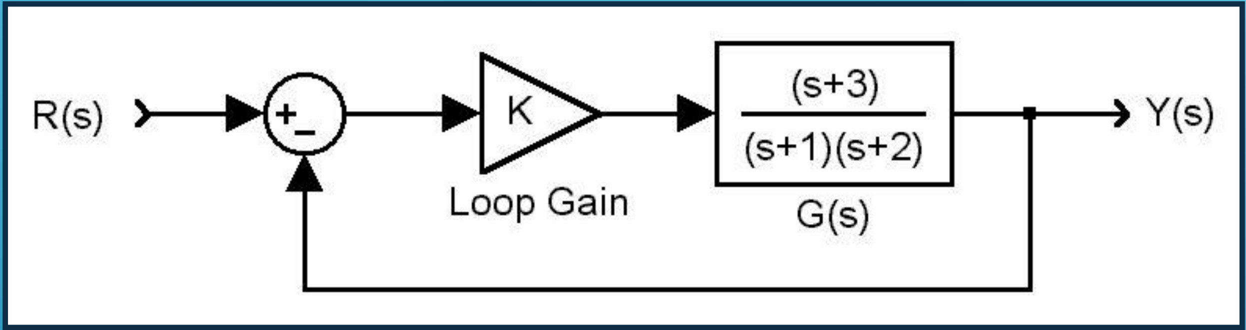
WEEK 9

This week's agenda

- Root Locus Analysis
- Design based on Root Locus
  - Lead Compensation
  - Lag Compensation
  - Lag-Lead Compensation
- Midterm

P-5 Root Locus Analysis

Consider the example below



$$T(s)=\frac{Y(s)}{R(s)}=\frac{K(s+3)}{s^2+(3+K)s+(2+3K)}$$

$$s_{1,2}=-\frac{3+K}{2}\pm\frac{1}{2}\sqrt{K^2-6K+1}$$

CL poles move as K changes!

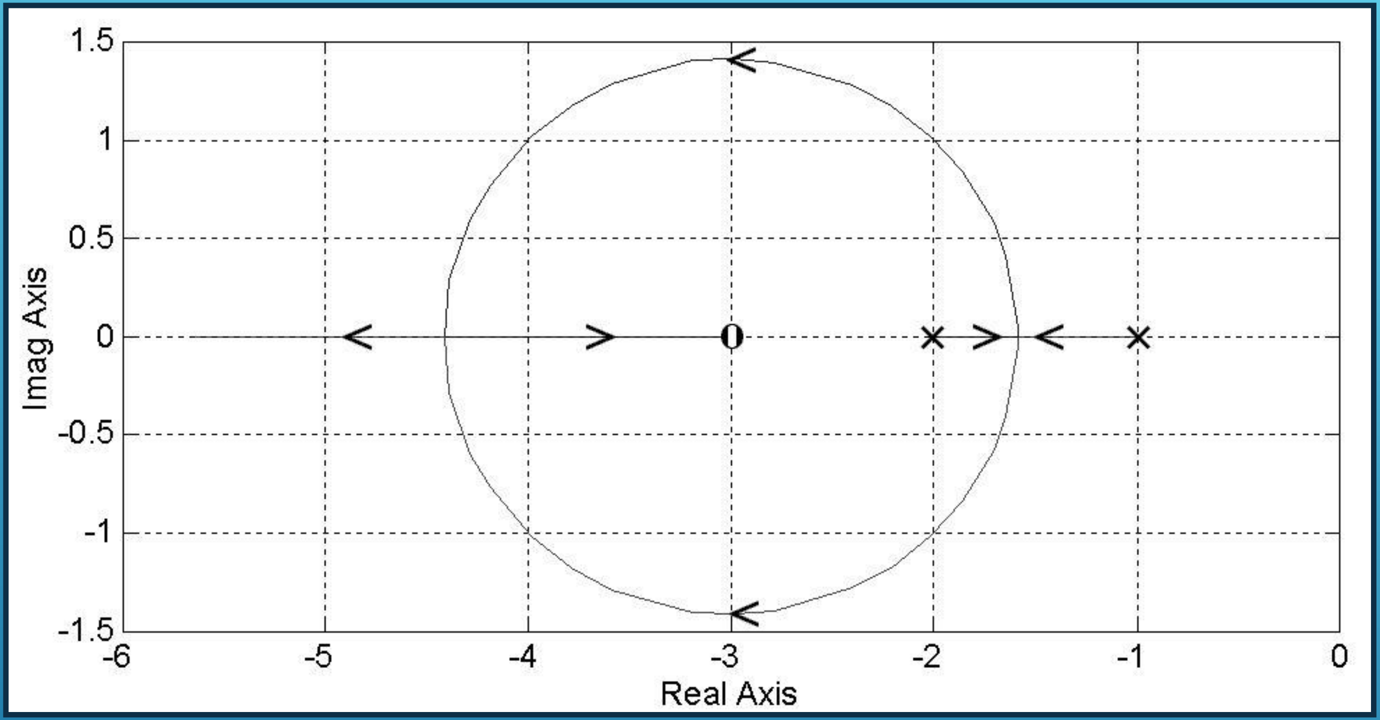
Root Locus Analysis

- Transient response characteristics are dependent upon the CL poles, and the CL poles move as the loop gain K changes.
- What happens if the design specifications require the CL poles at certain locations when the order of the denominator is more than 2? Difficult to repeat...
- What happens if the gain adjustment does not yield the desired result?

Root Locus Analysis

- **Root Locus Analysis will let us know how the CL poles change as the system parameters, e.g. gain, change.**
- **Root Locus Analysis is useful for finding approximate results very quickly.**

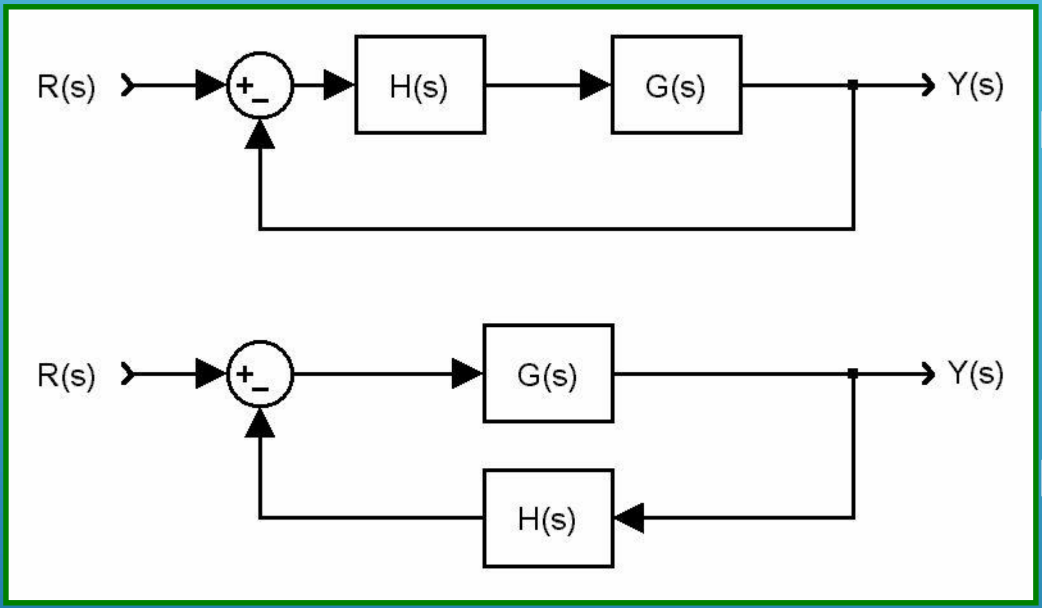
Root Locus Analysis
How to do with Matlab?



$$G(s) = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

```
» num=[1 3]
» den=[1 3 2]
» rlocus(num,den)
```

Root Locus Analysis
Angle and Magnitude Conditions



$$T(s) = \frac{G(s)H(s)}{\boxed{1 + G(s)H(s)}}$$

$$T(s) = \frac{G(s)}{\boxed{1 + G(s)H(s)}}$$

Characteristic polynomials are the same, so we need to analyze the locations of s satisfying G(s)H(s)=-1

Root Locus Analysis
Angle and Magnitude Conditions

$$s = \sigma + j\omega,\ s \text{ is a complex number}$$

$$G(\sigma + j\omega)H(\sigma + j\omega) = A(\sigma,\omega) + jB(\sigma,\omega)$$

$$|G(s)H(s)| = \sqrt{A(\sigma,\omega)^2 + B(\sigma,\omega)^2} = 1$$

$$\angle G(\sigma + j\omega)H(\sigma + j\omega) = \pm 180^\circ(2k+1)$$

Root Locus Analysis
Angle and Magnitude Conditions

In many cases, the characteristic equation can be written as

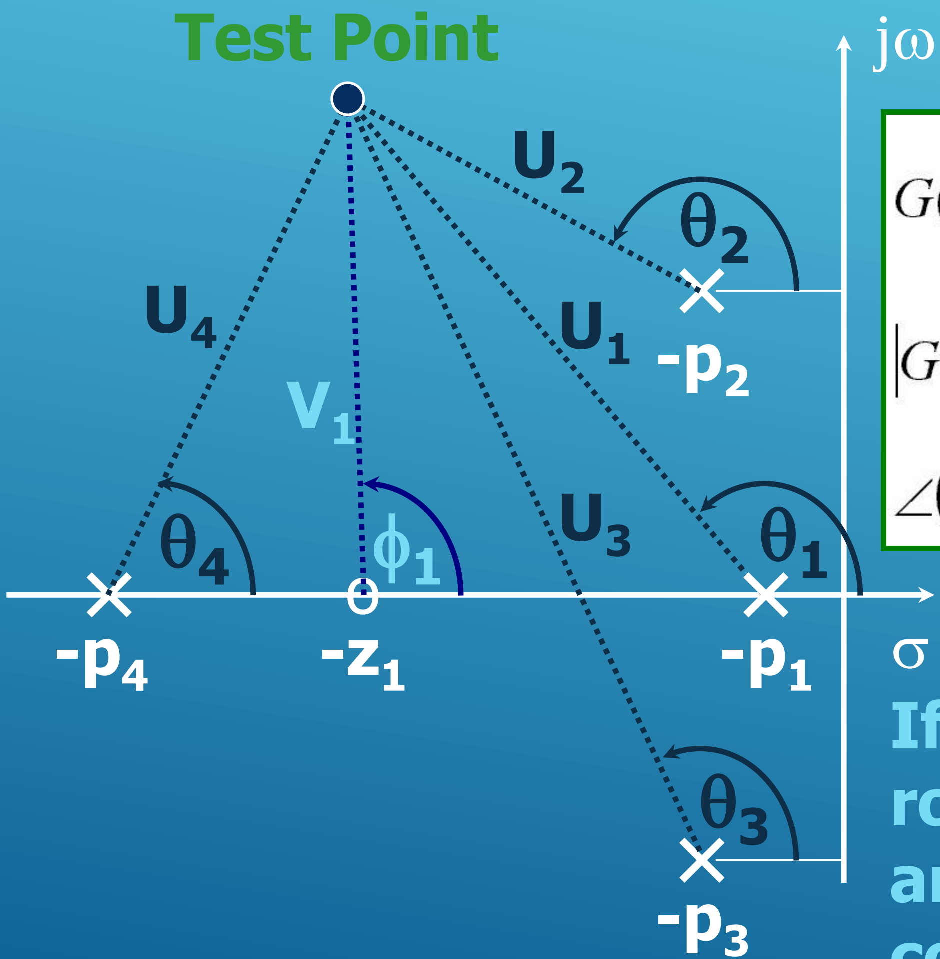
$$1+\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=0$$

Set s=s$_t$, a test point, and check the conditions.

$$\left|\frac{K(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right|\overset{?}{=}1$$

$$\angle\left(\frac{(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right)=\sum_{i=1}^{m}\phi_i-\sum_{i=1}^{n}\theta_i\overset{?}{=}\pm 180°(2k+1)$$

Root Locus Analysis
Angle and Magnitude Conditions - Example



$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

$$\left|G(s)H(s)\right| = \frac{KV_1}{U_1U_2U_3U_4}$$

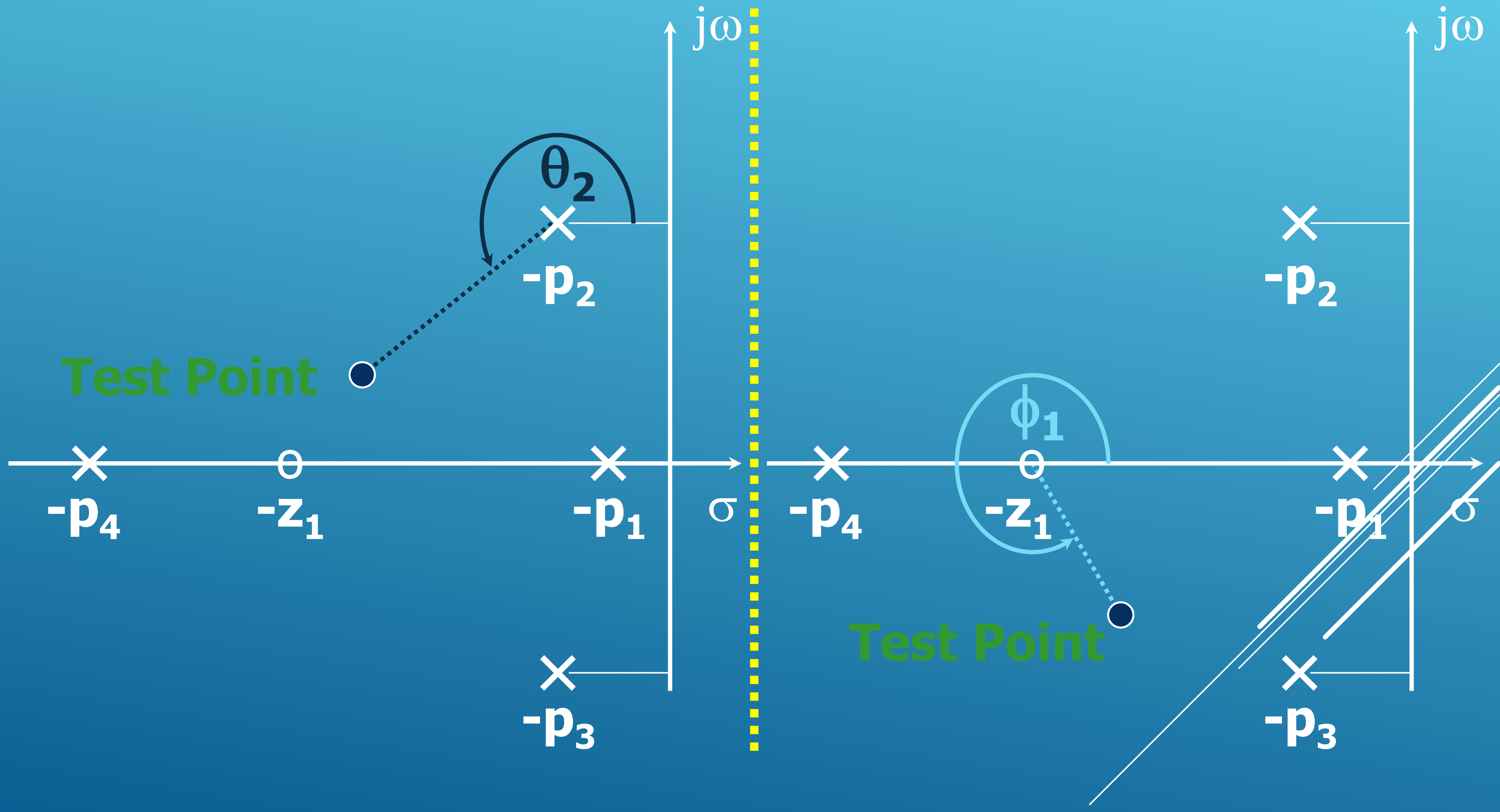
$$\angle\left(G(s)H(s)\right) = \phi_1 - \left(\theta_1 + \theta_2 + \theta_3 + \theta_4\right)$$

If the test point is on the root locus, it will satisfy the angle and magnitude conditions

Root Locus Analysis

Pay attention to the angle measurements!

Counter clockwise direction



Root Locus Analysis
Rules for Constructing Root Loci

1. Locate the open loop poles and zeros
2. Determine the loci on the real axis
3. Determine the asymptotes of root loci
4. Find the breakaway and break-in points
5. Determine the angle of departure from a complex pole
6. Determine the angle of arrival at a complex zero
7. Find the point where the root loci may cross the imaginary axis
8. Determine the shape of the root loci in the broad neighborhood of the $j\omega$ axis and the origin of the s-plane
9. Determine the closed loop poles

Root Locus Analysis - Rules

1. Locate the open loop poles and zeros

The root locus branches start from the open loop poles and terminate at zeros (finite zeros or zeros at infinity)

$$G(s)H(s) = K\frac{s+1}{s+2}$$

One finite pole at s=-2
One finite zero at s=-1

$$G(s)H(s) = K\frac{s+1}{(s+2)(s+3)}$$

Two finite poles at s=-2 and s=-3
One finite zero at s=-1
One zero at infinity

$$G(s)H(s) = K\frac{1}{s+2}$$

One finite pole at s=-2
One zero at infinity

Root Locus Analysis - Rules

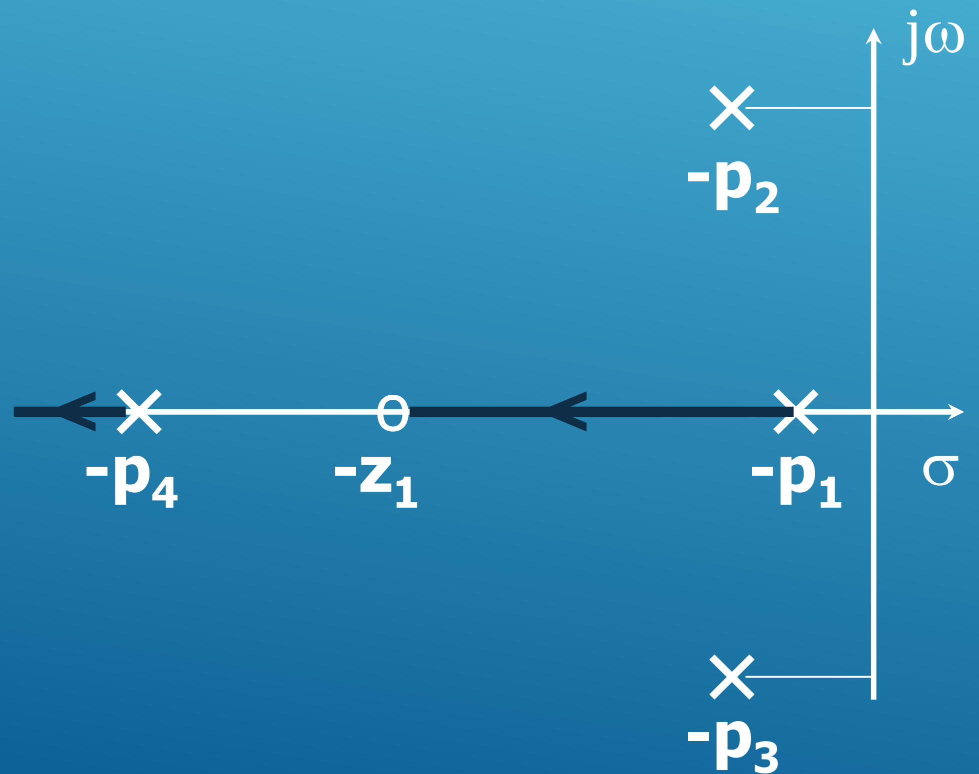
1. Locate the open loop poles and zeros

There are three zeros at infinity. The procedure will tell you where they are…

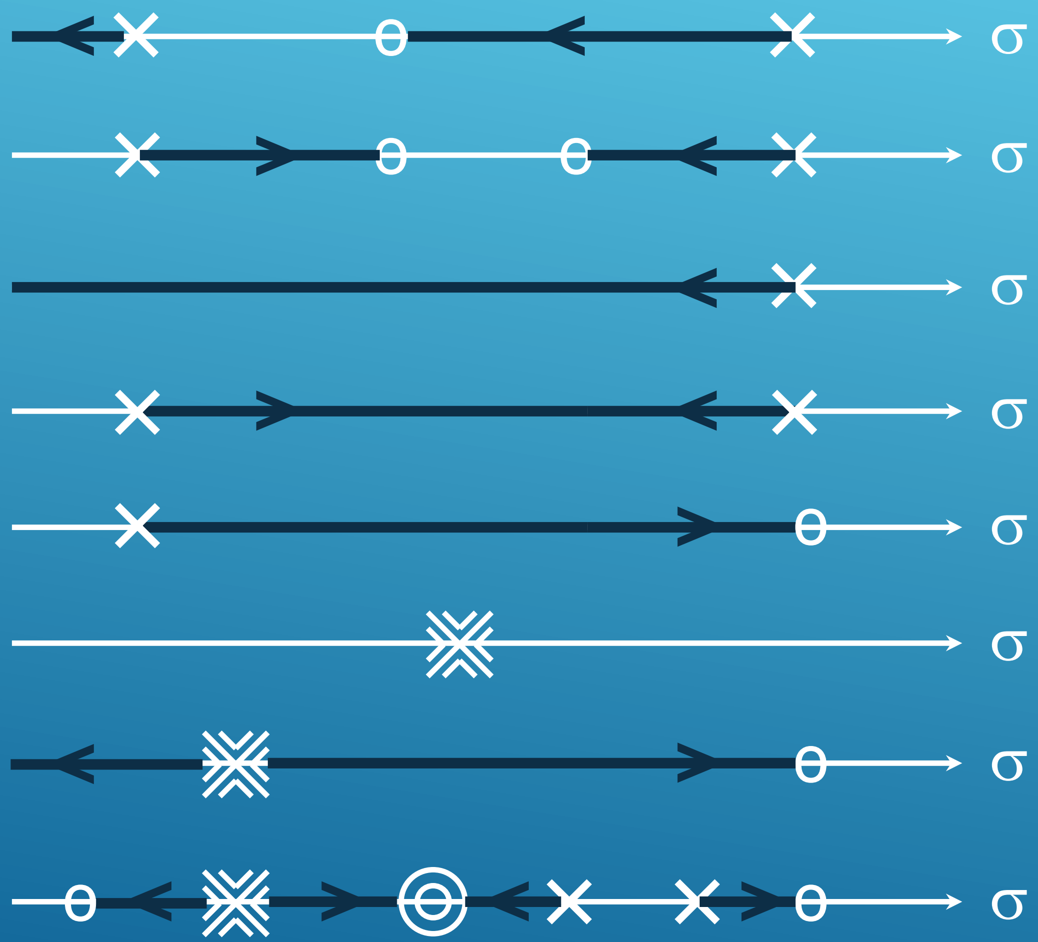
Root Locus Analysis - Rules

2. Determine the loci on the real axis

Consider only the poles and the zeros lying on the real axis. Choose a test point, if the number of poles and zeros right to the test point is odd, then the test point belongs to the root locus.



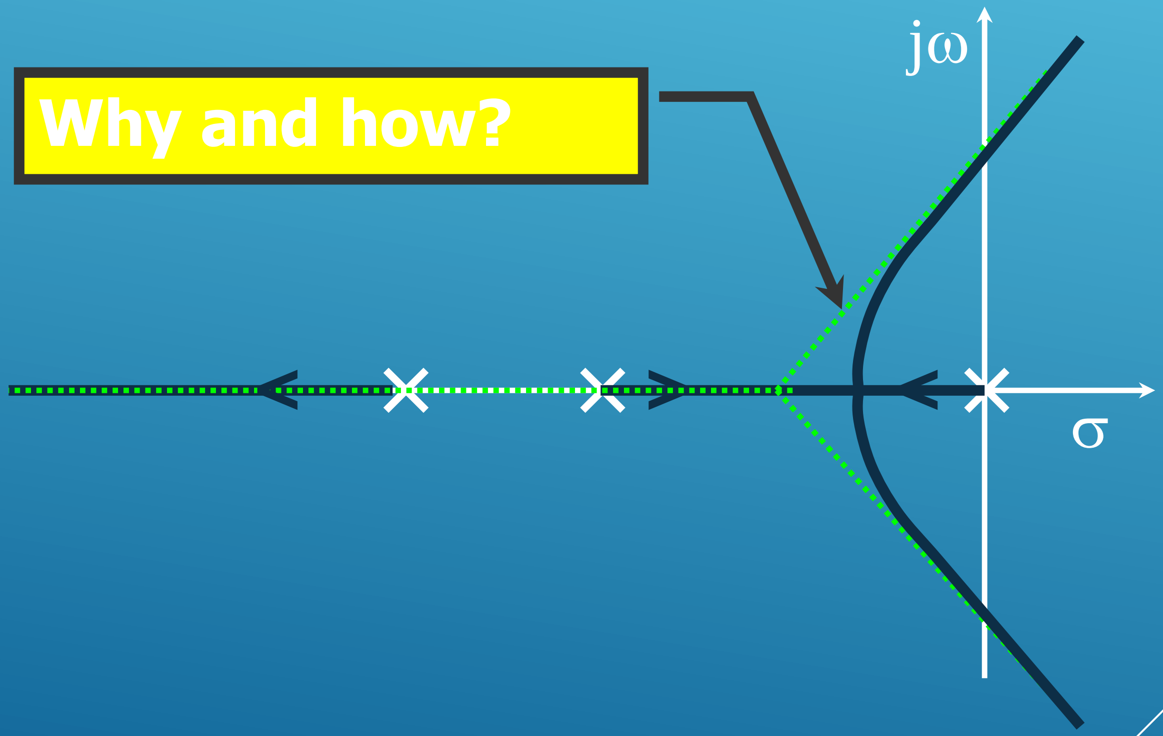
Root Locus Analysis - Rules
2. Determine the loci on the real axis



Root Locus Analysis - Rules

3. Determine the asymptotes of root loci

If there are open loop zeros at infinity, how does the root locus approach them?



Root Locus Analysis - Rules

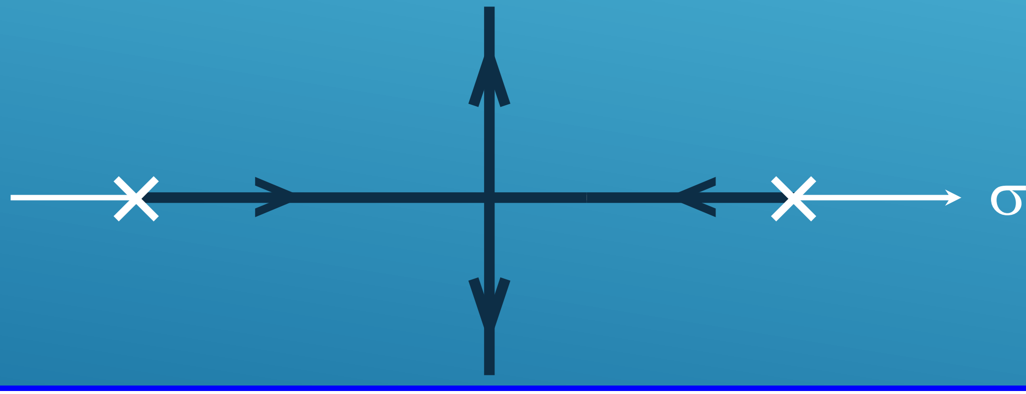
3. Determine the asymptotes of root loci

$$\text{Angle of asymptotes} = \frac{\pm 180^\circ(2k+1)}{n-m}, k = 0,1,2,\ldots$$

n : Number of poles of $G(s)H(s)$

m : Number of zeros of $G(s)H(s)$

Obviously finite poles and finite zeros!



$$\text{Angle of asymptotes} = \frac{\pm 180^\circ(2k+1)}{2-0}, k = 0,1,2,\ldots$$

$$\text{Angle of asymptotes} = \pm 90^\circ(2k+1), k = 0,1,2,\ldots$$

Root Locus Analysis - Rules

3. Determine the asymptotes of root loci

- **There are only *n-m* distinct asymptotes**
- **As *k* increases, the expression repeats itself**
- **Asymptotes intersect each other on the real axis since poles and zeros can occur in complex conjugate pairs**

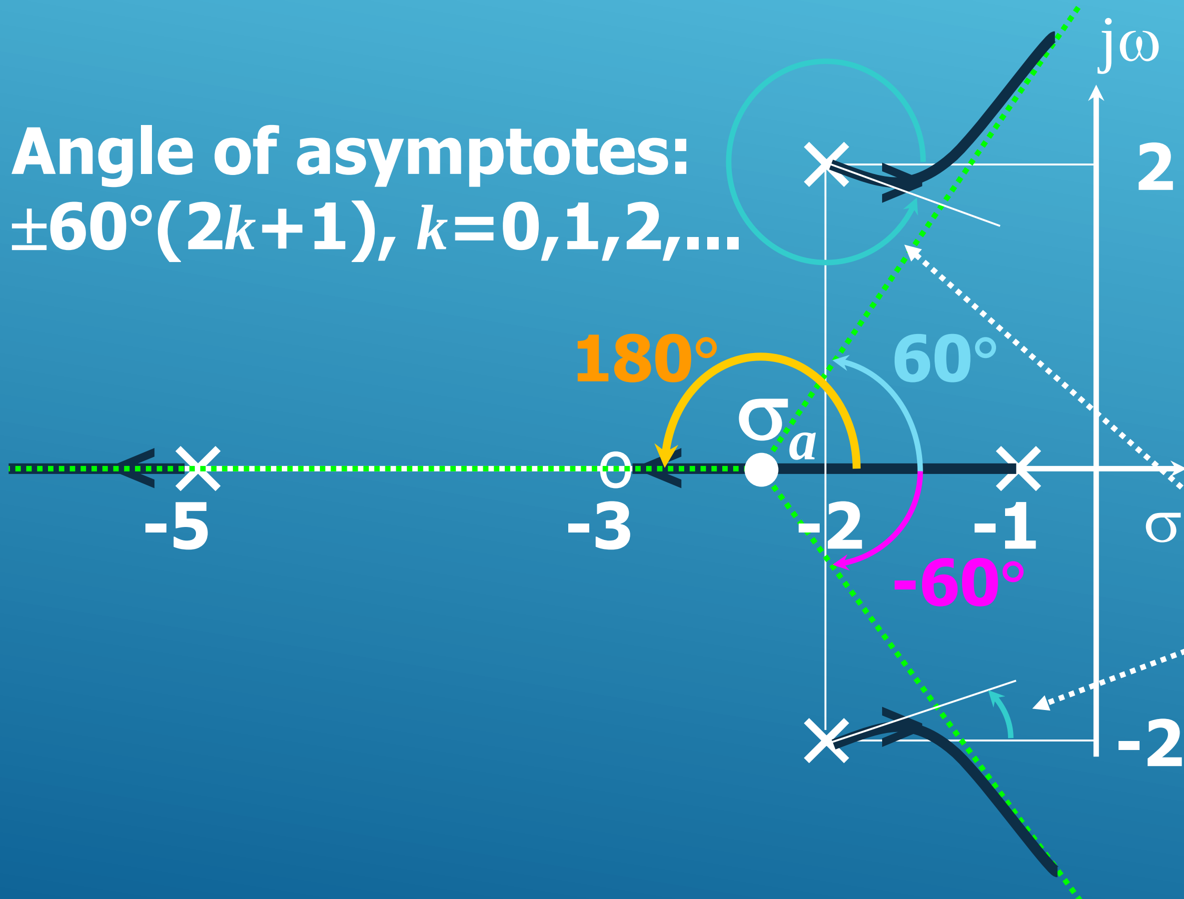
$$\text{Intersection Point } \sigma_a = \frac{(\text{sum of poles}) - (\text{sum of zeros})}{n - m}$$

Root Locus Analysis - Rules

3. Determine the asymptotes of root loci

An Example

Angle of asymptotes:
±60°(2k+1), k=0,1,2,...



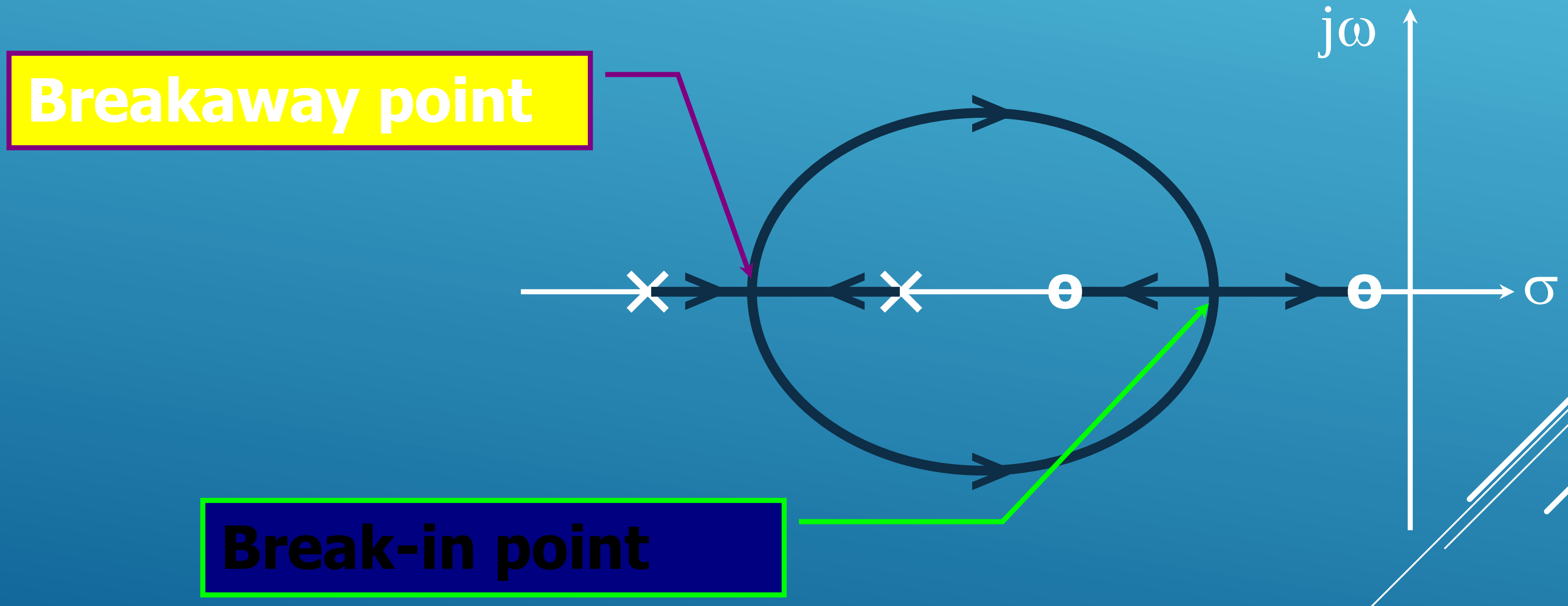
n=4, m=1
n-m=3
Σpoles=-10
Σzeros=-3
σ_a=-7/3

We will see how to find these angles

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

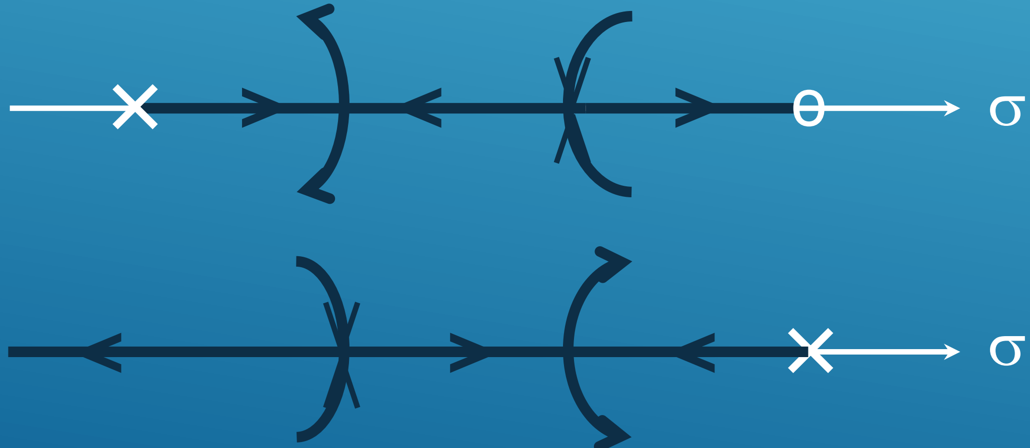
When two poles meet, breakaway point occurs. Similarly, if they tend to approach two zeros, they meet at break-in point.



Root Locus Analysis - Rules

4. Find the breakaway and break-in points

- Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in complex conjugate pairs.
- Pay attention to the following cases! You do not have to have them in between two zeros and two poles...



Root Locus Analysis - Rules

4. Find the breakaway and break-in points

Write the characteristic equation as

$$A(s) + KB(s) = 0$$

K = -A(s)/B(s)

and find the roots of

$$\frac{dK}{ds} = -\frac{A'(s)B(s) - A(s)B'(s)}{B(s)^2} = 0$$

where

$$B'(s) = \frac{dB(s)}{ds}, A'(s) = \frac{dA(s)}{ds}$$

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

$$A(s) + KB(s) = 0$$

$$\frac{dK}{ds} = -\frac{A'(s)B(s) - A(s)B'(s)}{B(s)^2} = 0$$

Solution of this equation will let you have a set of s values, say $\{s_1, s_2, \ldots, s_N\}$. Not all of them correspond to breakaway and break-in points. Some s_i values may not be on the root locus, then they do not correspond to breakaway or break-in points.

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

$$A(s) + KB(s) = 0$$

$$\frac{dK}{ds} = -\frac{A'(s)B(s) - A(s)B'(s)}{B(s)^2} = 0$$

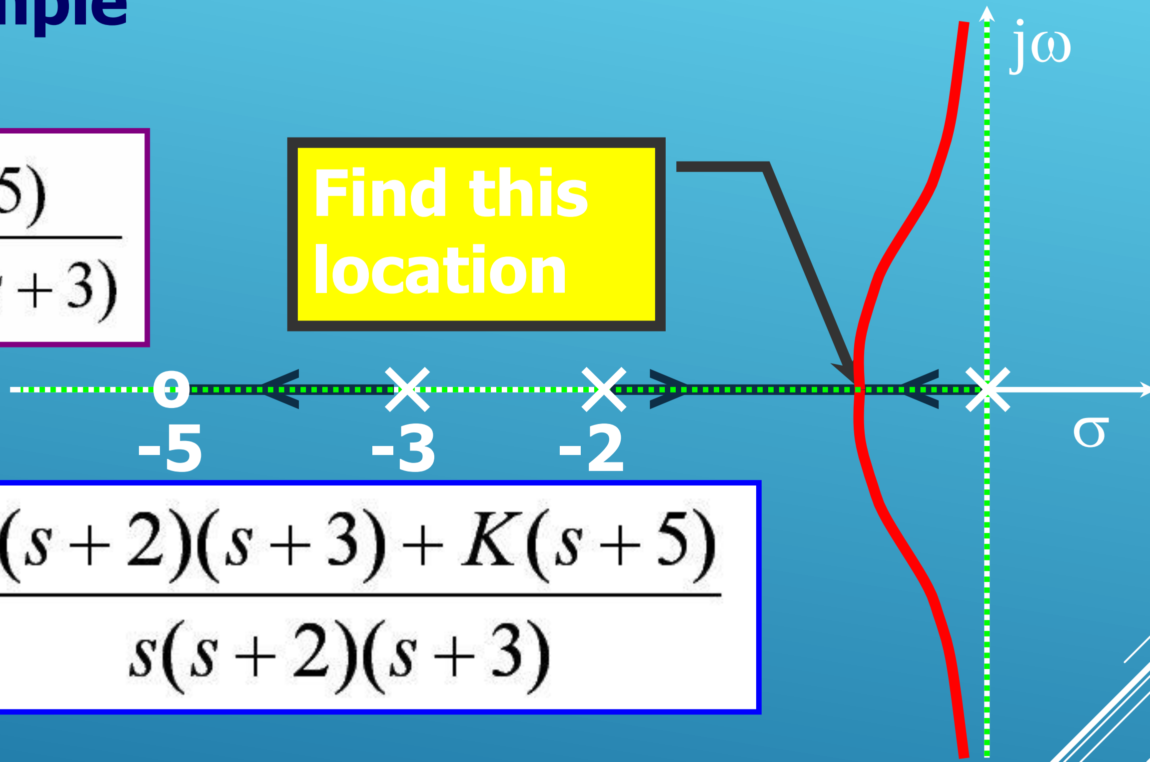
If s=s_i and s=s_j are complex conjugate pairs satisfying dK(s)/ds=0, and if you are not sure if these are on the root loci as breakaway or break-in points, calculate K and see if K≥0. If not, then these are not on the root loci!

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

An Example

$$G(s)H(s) = \frac{K(s+5)}{s(s+2)(s+3)}$$



$$1 + KG(s)H(s) = \frac{s(s+2)(s+3) + K(s+5)}{s(s+2)(s+3)}$$

$A(s)$ $B(s)$

$$s(s+2)(s+3) + K(s+5) = 0$$

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

An Example

$$A(s) = s(s+2)(s+3) = s^3 + 5s^2 + 6s$$

$$B(s) = (s+5)$$

$$\frac{dK}{ds} = \text{[redacted]}\frac{B'(s)A(s) - B(s)A'(s)}{(s+5)^2} = 0$$

$$= \text{[redacted]}\frac{1(s^3 + 5s^2 + 6s) - (s+5)(3s^2 + 10s + 6)}{(s+5)^2} = 0$$

$$= \text{[redacted]}\frac{(s^3 + 5s^2 + 6s) - (3s^3 + 25s^2 + 56s + 30)}{(s+5)^2} = 0$$

$$= \frac{2s^3 + 20s^2 + 50s + 30}{(s+5)^2} = 0 \Rightarrow$$

$s_1 = -0.8865$

$s_2 = -2.5964$

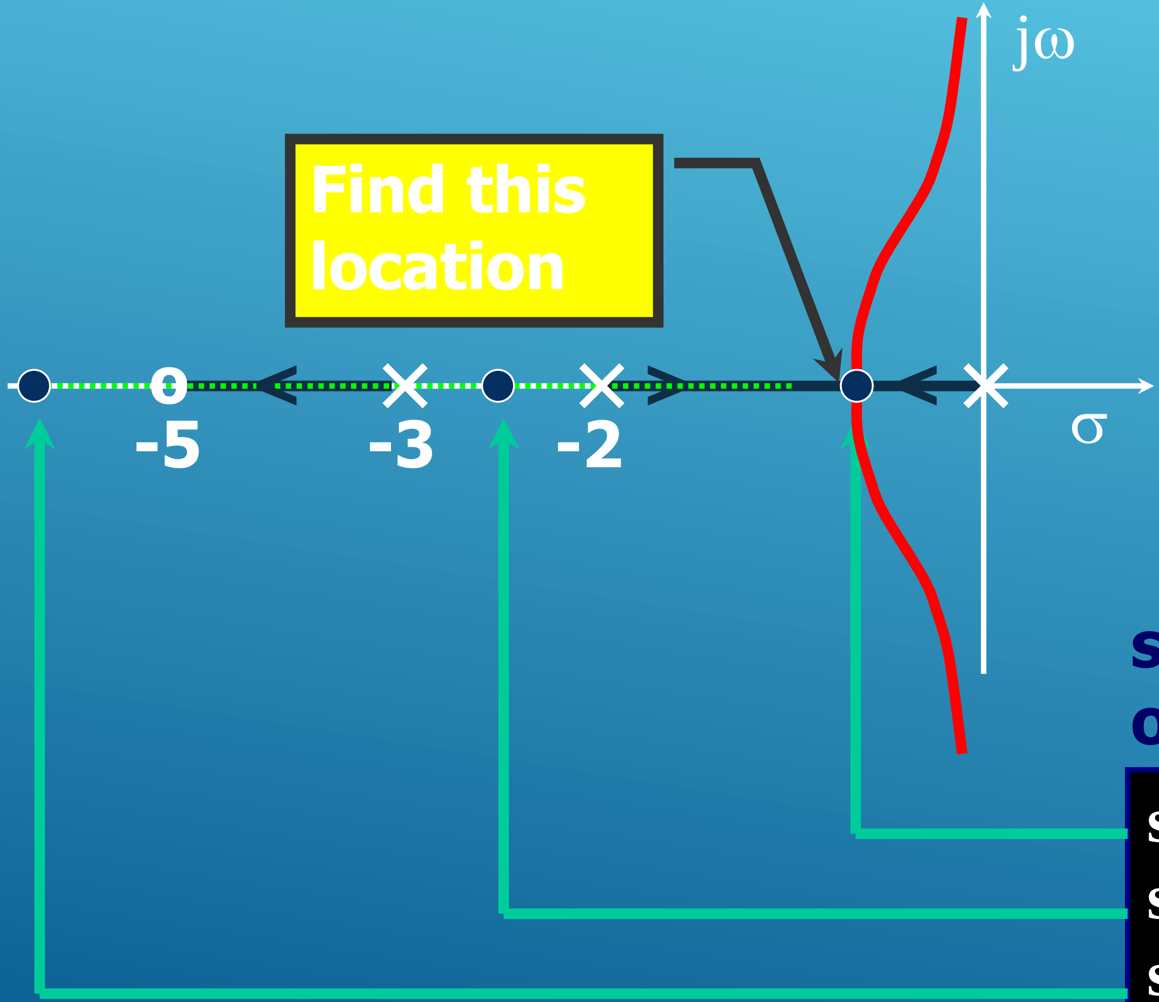
$s_3 = -6.5171$

[redacted]

Root Locus Analysis - Rules

4. Find the breakaway and break-in points

An Example

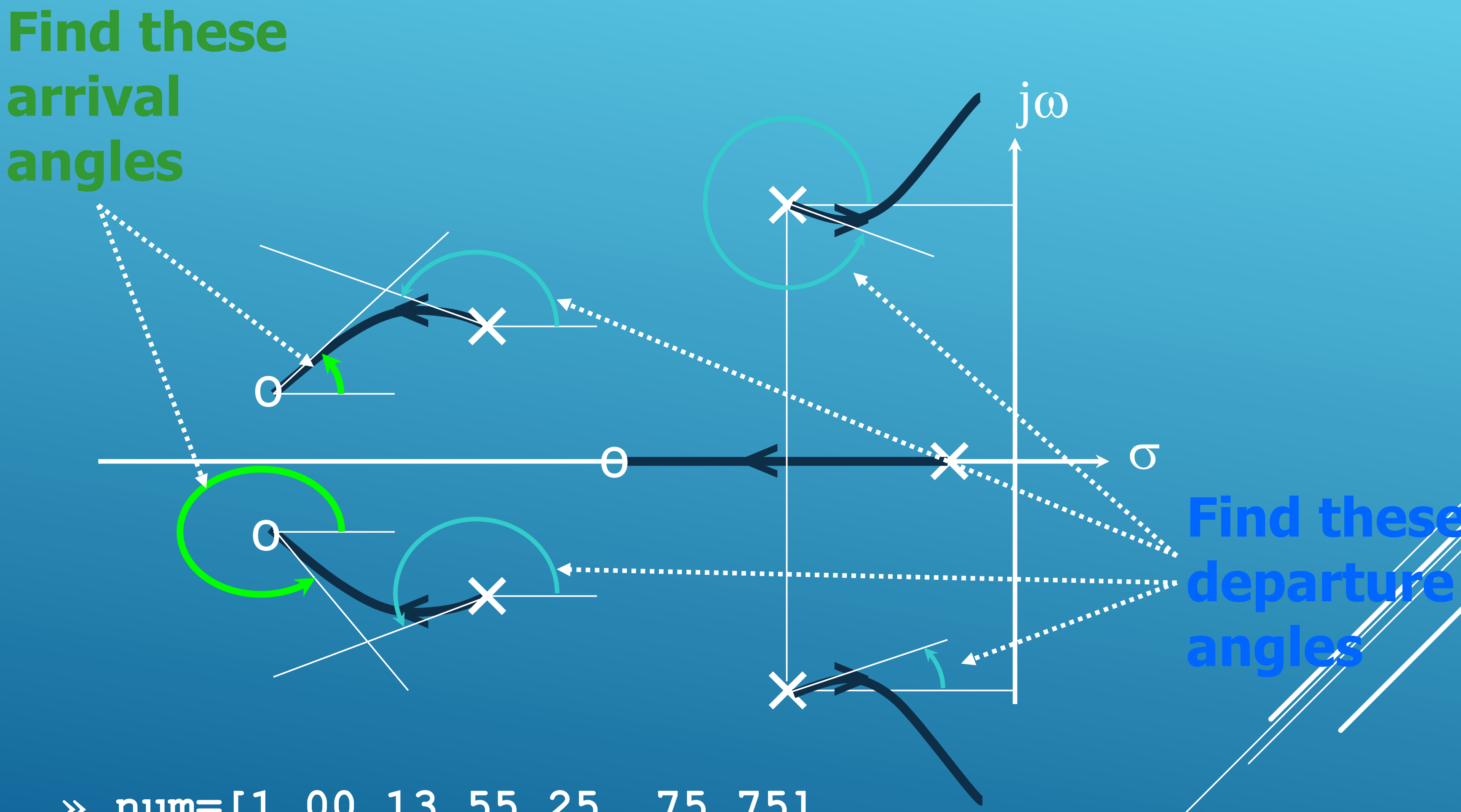


s_2 and s_3 are not on the root loci

$s_1 = -0.8865$
$s_2 = -2.5964$
$s_3 = -6.5171$

Root Locus Analysis - Rules

5-6. Determine angle of departure/arrival




```
» num=[1.00 13 55.25  75.75]
» den=[1.00 12 59.50 153.25 218.8125 114.0625]
» rlocus(num,den)
```

Root Locus Analysis - Rules
5. Determine angle of departure

Angle (α) of departure from a complex pole is

$\alpha = 180° - \Sigma$(Angles from other poles to that pole)
$+\Sigma$(Angles from zeros to that pole)

Remember the angle condition

$$\left|\frac{K(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right| \overset{?}{=} 1$$

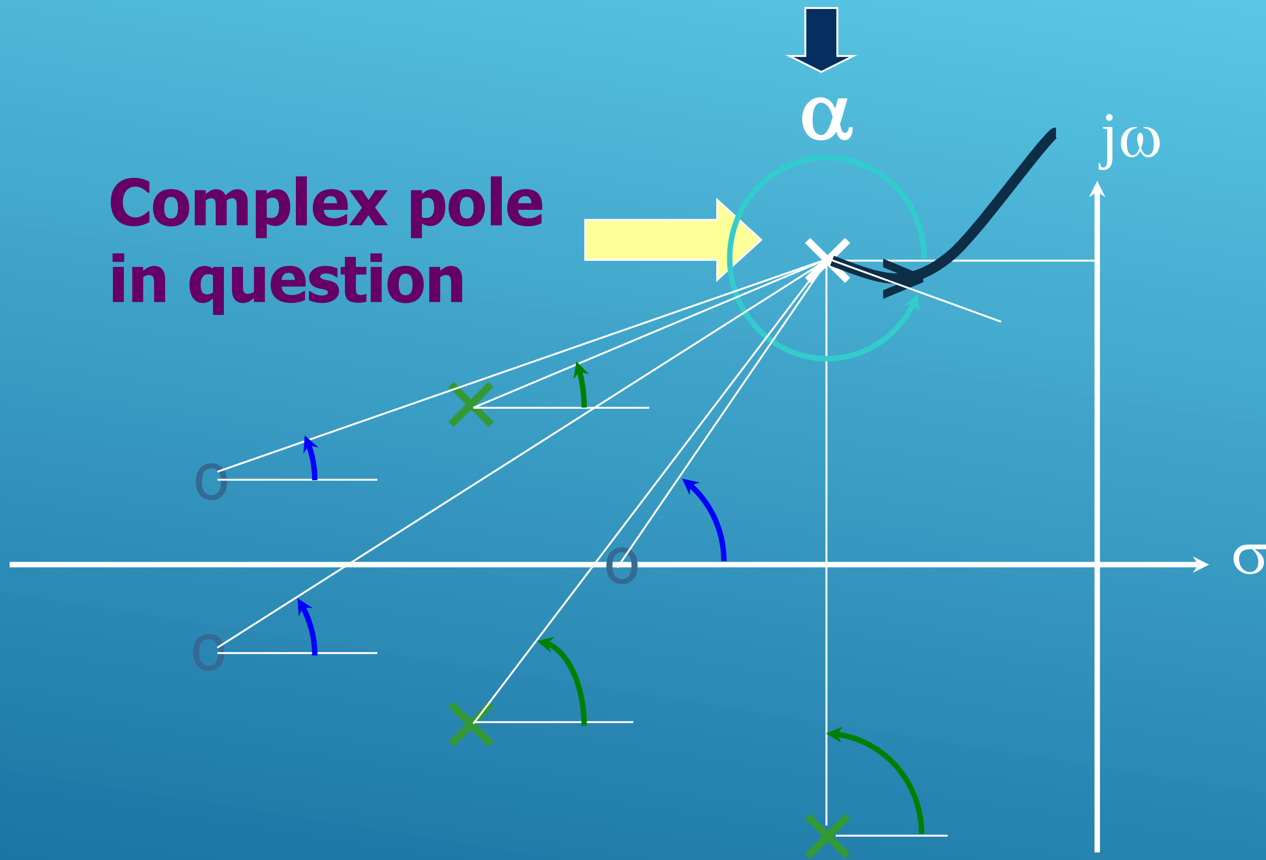
From zeros

From poles

$$\angle\left(\frac{(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right) = \sum_{i=1}^{m}\phi_i - \sum_{i=1}^{n}\theta_i \overset{?}{=} \pm 180°(2k+1)$$

Root Locus Analysis - Rules

5. Determine angle of departure



$$\alpha = 180^\circ - \Sigma(\text{Angles from other poles to that pole}) + \Sigma(\text{Angles from zeros to that pole})$$

Root Locus Analysis - Rules

6. Determine angle of arrival

Angle (β) of arrival at a complex zero is

β = 180° − Σ(Angles from other zeros to that zero) + Σ(Angles from poles to that zero)

Remember the angle condition

$$\left|\frac{K(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right| \overset{?}{=} 1$$

From zeros

From poles

$$\angle\left(\frac{(s_t+z_1)(s_t+z_2)\cdots(s_t+z_m)}{(s_t+p_1)(s_t+p_2)\cdots(s_t+p_n)}\right) = \sum_{i=1}^{m}\phi_i - \sum_{i=1}^{n}\theta_i \overset{?}{=} \pm 180^\circ(2k+1)$$

Root Locus Analysis - Rules

6. Determine angle of arrival

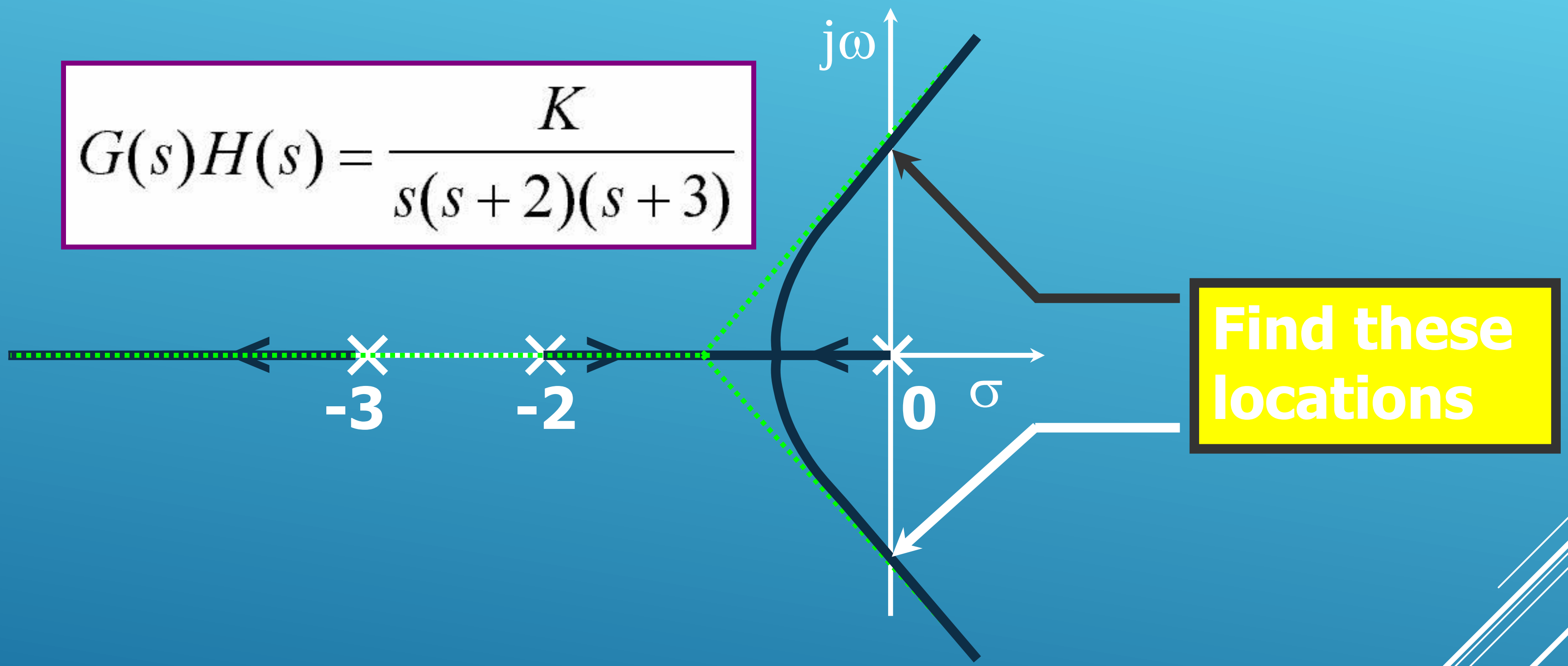
Complex zero in question

$$\beta = 180^\circ - \Sigma(\text{Angles from other zeros to that zero}) + \Sigma(\text{Angles from poles to that zero})$$

Root Locus Analysis - Rules

7. Find the jω axis crossings

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$



1. Use Routh's stability criterion to find critical K
2. In the characteristic equation, insert s=jω, and equate both real and imaginary part to zero, and solve for ω and K.

Root Locus Analysis - Rules

7. Find the jω axis crossings

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$

Open loop TF

$$s^3 + 5s^2 + 6s + K = 0$$

Characteristic equation

s^3	1	6
s^2	5	K
s^1	$6-0.2K$	0
s^0	K	

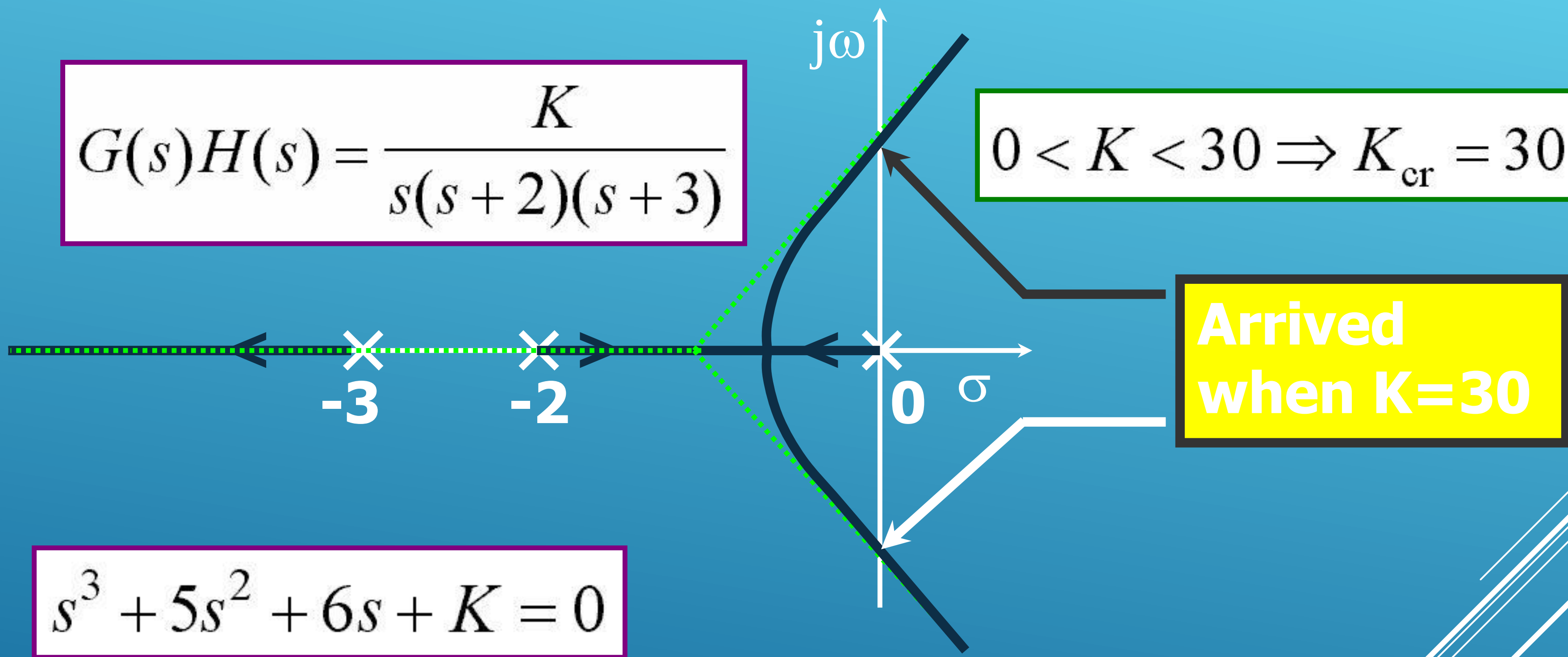
Routh table

$$0 < K < 30 \Rightarrow K_{cr} = 30$$

Root Locus Analysis - Rules

7. Find the jω axis crossings

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$



$$0 < K < 30 \Rightarrow K_{cr} = 30$$

$$s^3 + 5s^2 + 6s + K = 0$$

Remember, when K=0, the roots of the characteristic equation are the open loop poles, and with K=0 there is a pole at origin (on the imag. axis)

Root Locus Analysis - Rules

7. Find the jω axis crossings

$$s^3 + 5s^2 + 6s + K = 0$$

Insert s= jω

$$(j\omega)^3 + 5(j\omega)^2 + 6(j\omega) + K = 0$$

$$-j\omega^3 - 5\omega^2 + j6\omega + K = 0$$

$$(K - 5\omega^2) - j\omega(\omega^2 - 6) = 0$$

$$K - 5\omega^2 = 0$$

$$\omega^2 - 6 = 0$$

$$\omega_{cr} = \pm\sqrt{K/5} \text{ rad/sec}$$

$$\omega_{cr} = \pm\sqrt{6} \text{ rad/sec}$$

K is obtained from Routh test

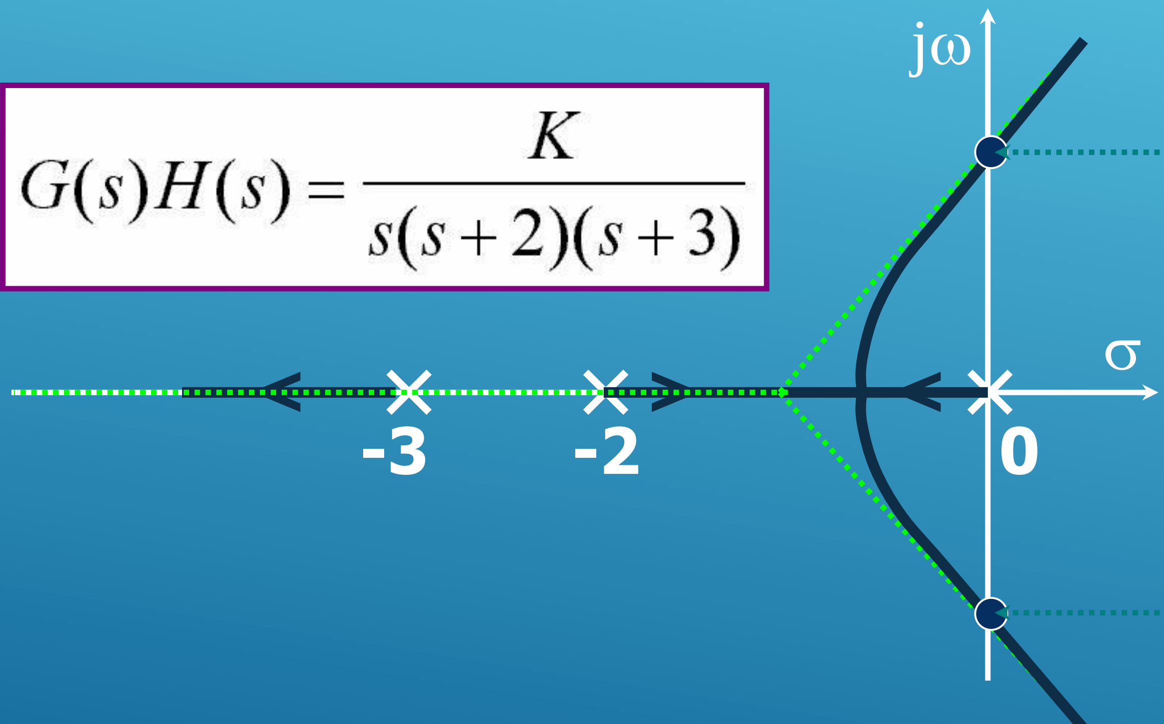
$$0 < K < 30 \Rightarrow K_{cr} = 30$$

Both yield the same result

Root Locus Analysis - Rules

7. Find the jω axis crossings

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$



$$K = K_{cr} = 30$$
$$\omega = \omega_{cr} = j\sqrt{6}$$

$$K = K_{cr} = 30$$
$$\omega = \omega_{cr} = -j\sqrt{6}$$

Root Locus Analysis - Rules

8. Focus on the important parts of the loci

Near origin behavior and the behavior around the imaginary axis must be well known.

Do your computational trials with high accuracy when the locus is around the imaginary axis.

Root Locus Analysis - Rules
9. Determine the closed loop poles

Remember, once you set the value of K, this fixes locations of the CL poles. This is because the magnitude condition is satisfied on the root loci.

$$1+KG(s)H(s)=0 \Rightarrow KG(s)H(s)=-1$$

$$\Rightarrow |KG(s)H(s)|=1 \Rightarrow K|G(s)H(s)|=1$$

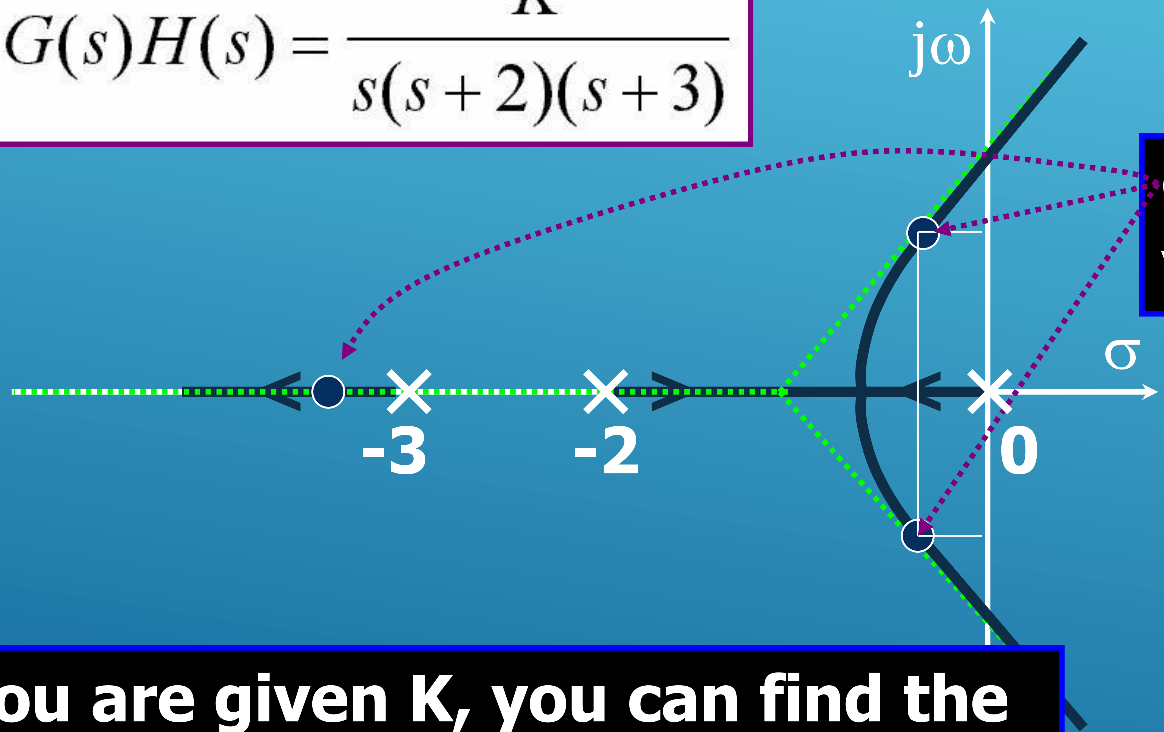
$$|G(s)H(s)|=\frac{1}{K} \text{ or } K=\frac{1}{|G(s)H(s)|}$$

→ **Given CL poles, find K**

Root Locus Analysis - Rules

9. Determine the closed loop poles

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$



Given CL poles, you can find K

If you are given K, you can find the CL poles from the characteristic equation

$$s^3 + 5s^2 + 6s + K = 0$$

Root Locus Analysis - Rules

9. Determine the closed loop poles

Look at the magnitude condition

$$G(s)H(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$\left|G(s)H(s)\right| = \frac{|s+z_1||s+z_2|\cdots|s+z_m|}{|s+p_1||s+p_2|\cdots|s+p_n|} = \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{i=1}^{n}|s+p_i|} = \frac{1}{K}$$

$$K = \frac{\prod_{i=1}^{n}|s+p_i|}{\prod_{i=1}^{m}|s+z_i|}$$

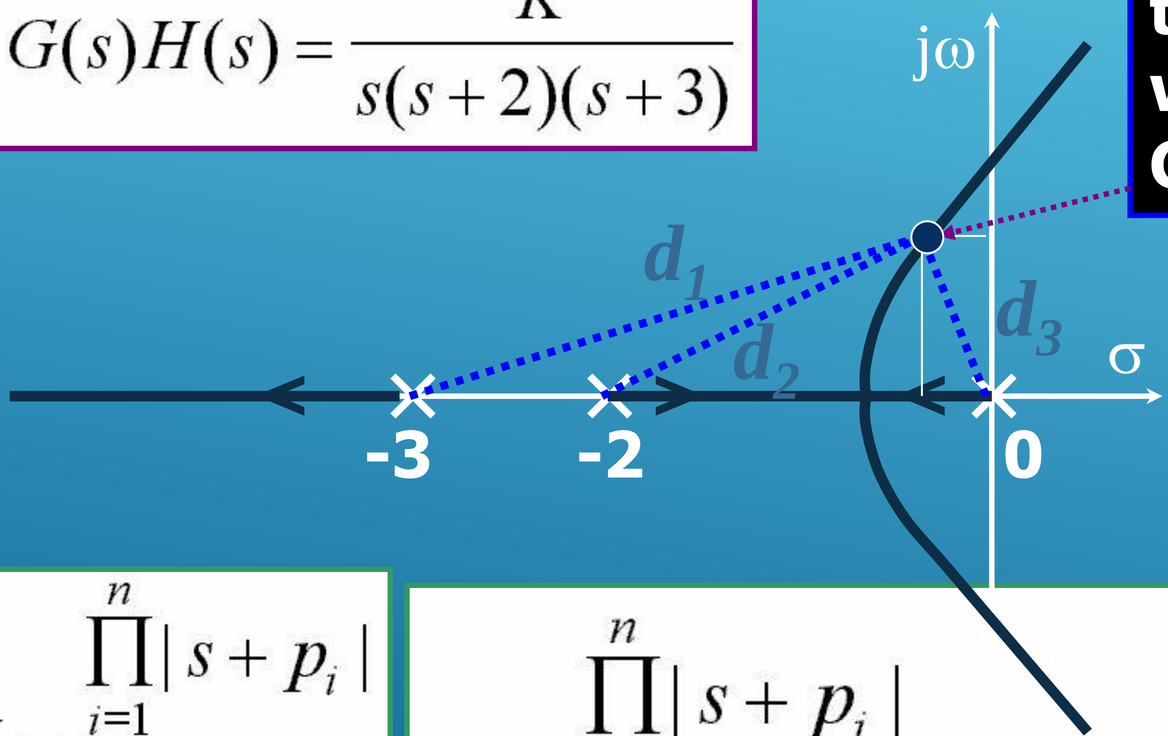
ple

Root Locus Analysis - Rules

9. Determine the closed loop poles

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)}$$

Assume that this point is wanted to be a CL pole



$$K = \frac{\prod_{i=1}^{n} |s + p_i|}{\prod_{i=1}^{m} |s + z_i|}$$

$$K = \frac{\prod_{i=1}^{n} |s + p_i|}{1} = d_1 d_2 d_3$$