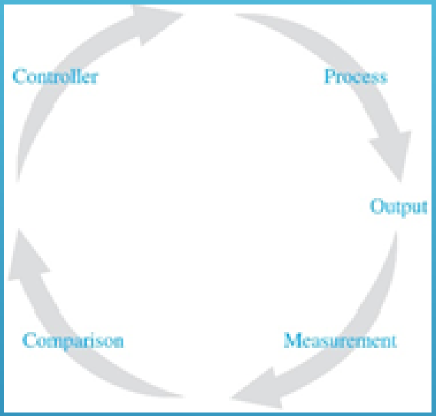
CONTROL SYSTEMS



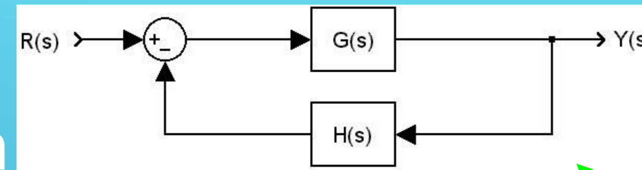
Doç. Dr. Murat Efe

WEEK 12

Root Locus Analysis - Summary
Rules for Constructing Root Loci

1. Locate the open loop poles and zeros
2. Determine the loci on the real axis
3. Determine the asymptotes of root loci
4. Find the breakaway and break-in points
5. Determine the angle of departure from a complex pole
6. Determine the angle of arrival at a complex zero
7. Find the point where the root loci may cross the imaginary axis
8. Determine the shape of the root loci in the broad neighborhood of the $j\omega$ axis and the origin of the s-plane
9. Determine the closed loop poles

Root Locus Analysis
Pole-Zero Cancellation



$$G(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{D'(s)(s+\alpha)} \text{ and } H(s)=\frac{A(s)}{B(s)}=\frac{A'(s)(s+\alpha)}{B(s)}$$

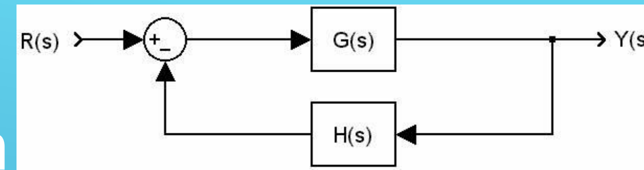
$$\boxed{T(s)=\frac{G(s)}{1+G(s)H(s)}}=\frac{\dfrac{N(s)}{D'(s)(s+\alpha)}}{1+\dfrac{N(s)}{D'(s)(s+\alpha)}\dfrac{A'(s)(s+\alpha)}{B(s)}}$$

$$=\frac{\dfrac{N(s)}{D'(s)(s+\alpha)}\dfrac{B(s)}{B(s)}}{\dfrac{D'(s)B(s)+N(s)A'(s)}{D'(s)B(s)}}=\frac{\dfrac{N(s)B(s)}{D'(s)B(s)}}{\dfrac{D'(s)B(s)+N(s)A'(s)}{D'(s)B(s)}}\frac{1}{(s+\alpha)}$$

$$=\frac{N(s)B(s)}{D'(s)B(s)+N(s)A'(s)}\frac{1}{(s+\alpha)}$$

Canceled pole of G(s) is kept as a CL pole!

Root Locus Analysis Pole-Zero Cancellation An Example



$$G(s) = \frac{s+1}{(s+2)(s+3)}, \text{ and } H(s) = K\frac{s+3}{s+4}$$

(s+3) is common

$$1 + KG(s)H(s) = 1 + \frac{K(s+1)}{(s+2)(s+4)}\left(\frac{s+3}{s+3}\right)$$

(s+3) terms cancel

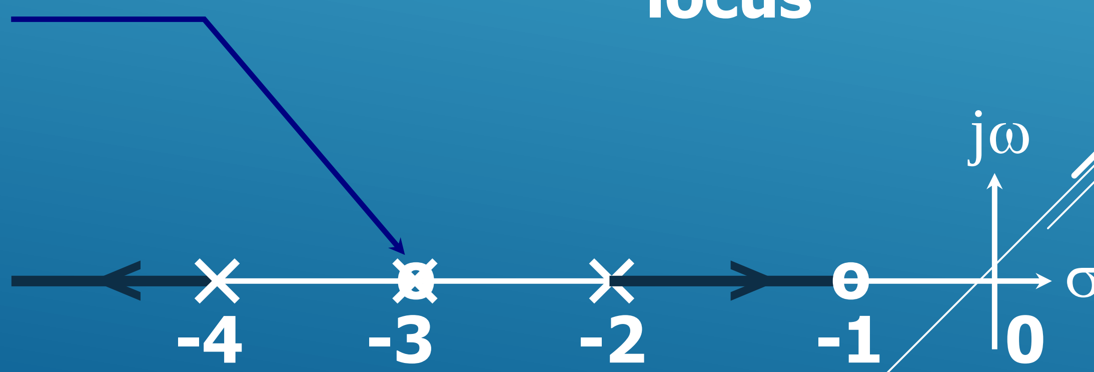
$$\big((s+2)(s+4) + K(s+1)\big)(s+3) = 0$$

Char. Eqn.

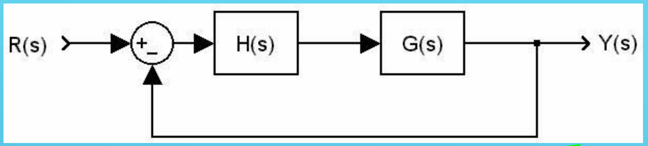
$$\big((s+2)(s+4) + K(s+1)\big) = 0$$

Char. Eqn. for root locus

Here is the pole-zero cancellation!



Root Locus Analysis
Pole-Zero Cancellation



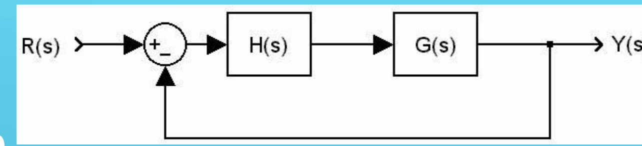
$$G(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{D'(s)(s+\alpha)} \text{ and } H(s)=\frac{A(s)}{B(s)}=\frac{A'(s)(s+\alpha)}{B(s)}$$

$$T(s)=\boxed{\frac{G(s)H(s)}{1+G(s)H(s)}}=\frac{\dfrac{N(s)}{D'(s)(s+\alpha)}\dfrac{A'(s)(s+\alpha)}{B(s)}}{1+\dfrac{N(s)}{D'(s)(s+\alpha)}\dfrac{A'(s)(s+\alpha)}{B(s)}}$$

Canceled pole is not a closed loop pole at all

$$=\frac{\dfrac{N(s)A'(s)}{D'(s)B(s)}}{\dfrac{D'(s)B(s)+N(s)A'(s)}{D'(s)B(s)}}=\frac{N(s)A'(s)}{D'(s)B(s)+N(s)A'(s)}$$

Root Locus Analysis
Pole-Zero Cancellation
An Example (Same result is obtained!)



$$G(s)=\frac{s+1}{(s+2)(s+3)},\ \text{and } H(s)=K\frac{s+3}{s+4}$$

- **(s+3) is common**

$$1+KG(s)H(s)=1+\frac{K(s+1)}{(s+2)(s+4)}\left(\frac{s+3}{s+3}\right)$$

- **(s+3) terms cancel**

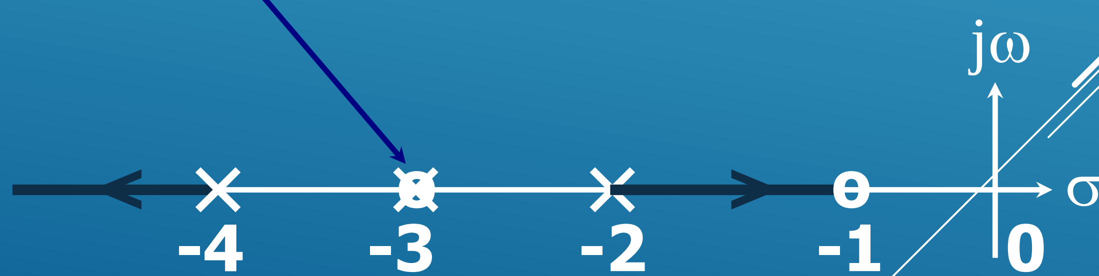
$$\big((s+2)(s+4)+K(s+1)\big)(s+3)=0$$

- **Char. Eqn.**

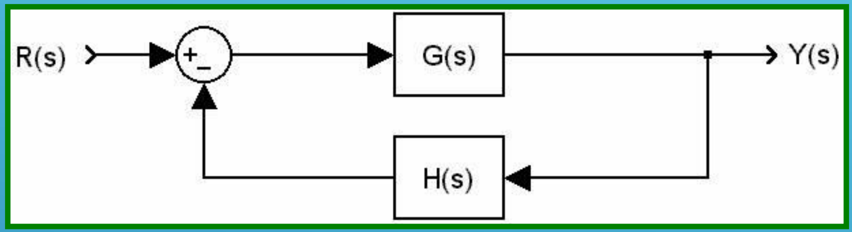
$$\big((s+2)(s+4)+K(s+1)\big)=0$$

- **Char. Eqn. for root locus**

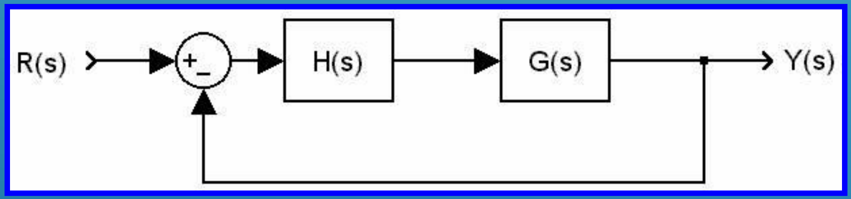
Here is the pole-zero cancellation!



Root Locus Analysis
Pole-Zero Cancellation



Canceled pole <u>is a CL</u> pole
Root locus does not notice it

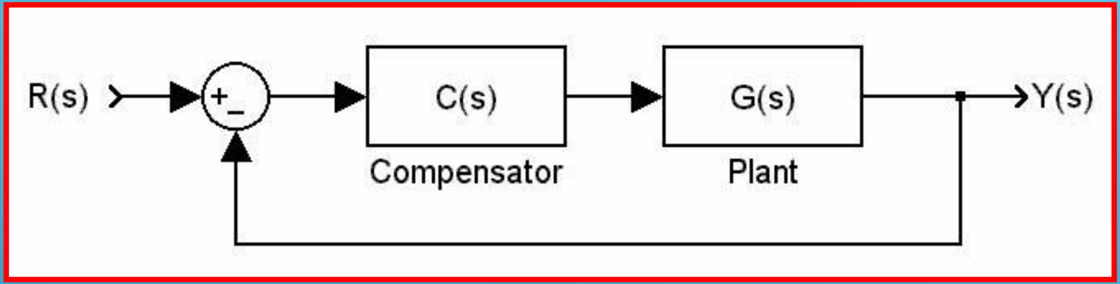


Canceled pole <u>is NOT a CL</u> pole
Root locus does not notice it

P-5 Design based on Root Locus

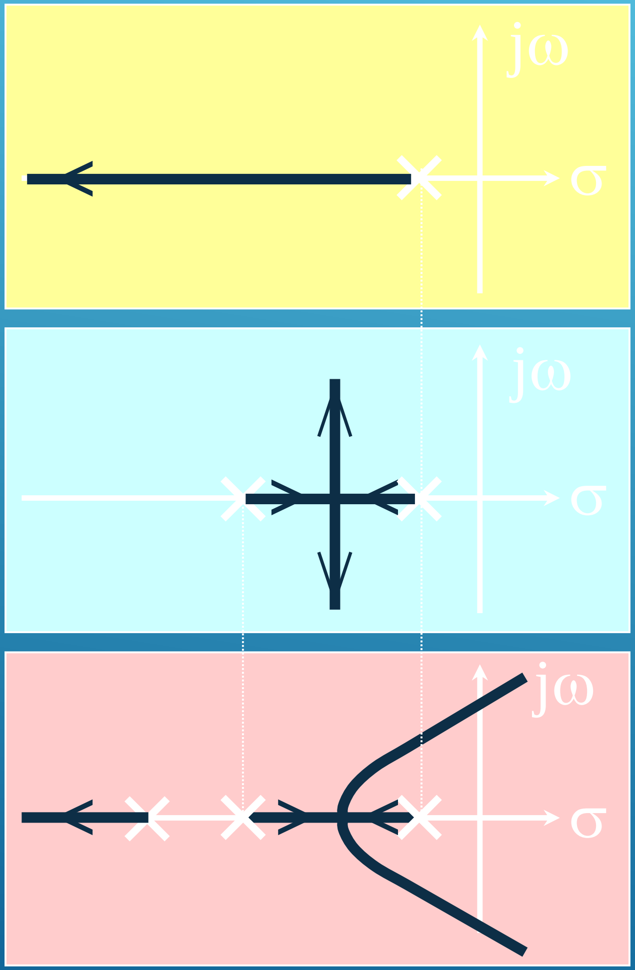
- **The goal is to meet the design specifications, and the way we followed so far has been to modify the gain K. What if this is not sufficient?**
- **Modify the system dynamics suitably to obtain the desired result, which means compensation, and the device you used is called compensator.**

Design based on Root Locus
Description of the Compensation Problem



- C(s) may remove some poles of G(s) and may add new poles, or C(s) may remove some zeros of G(s) and may add new zeros to change the shape of root locus.
- Once the shape of root locus becomes suitable to locate the desired closed loop poles, the adjustment of loop gain K is performed.

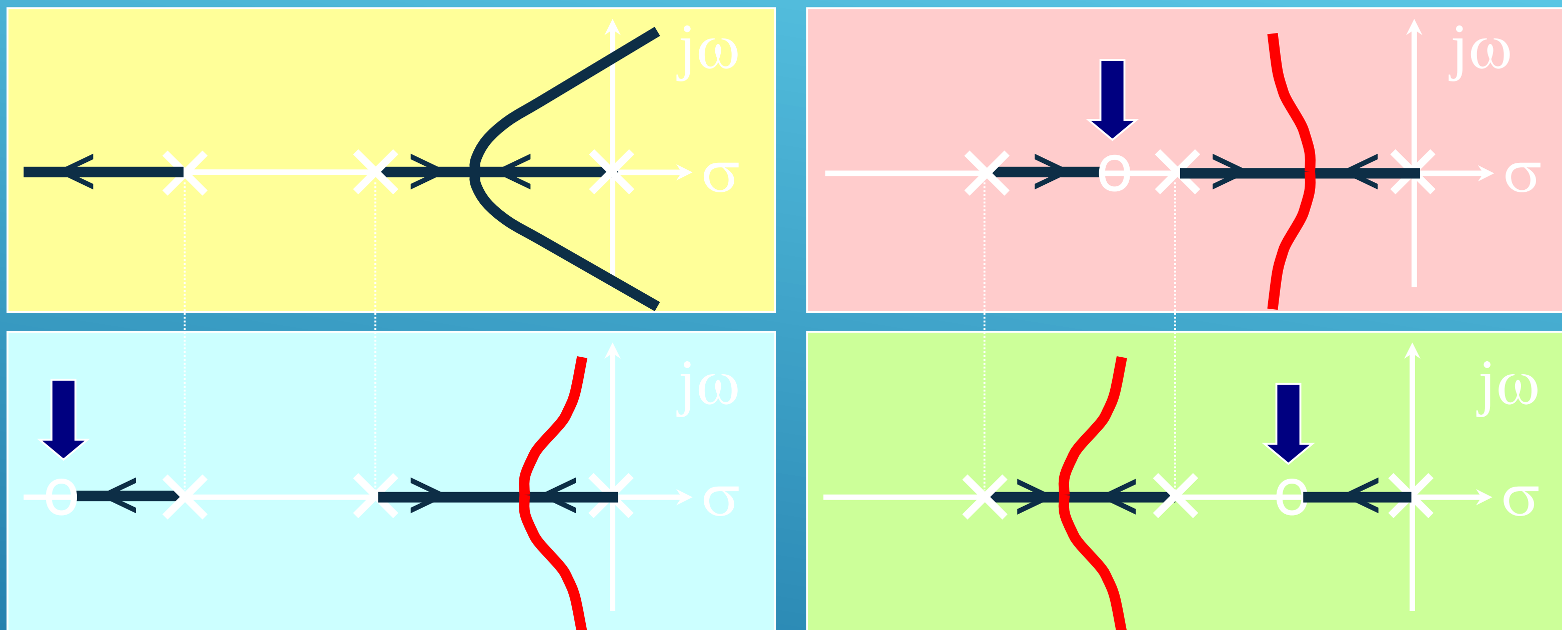
Design based on Root Locus
Effects of Addition of Poles



Adding poles pulls the root locus to the right

After some value of K, two of the CL poles are unstable!

Design based on Root Locus
Effects of Addition of Zeros

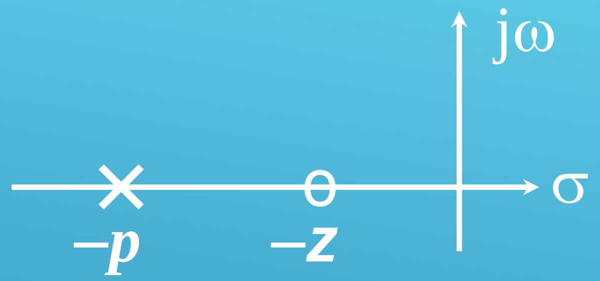


Adding zeros pulls the root locus to the left

Notice that, the CL poles are always stable for this example. Adding zeros increase the stability of the CL system, this is due to the anticipatory behavior of the derivative action.

Design based on Root Locus
Lead Compensation

$$C(s) = K\frac{s+z}{s+p},\ z < p$$

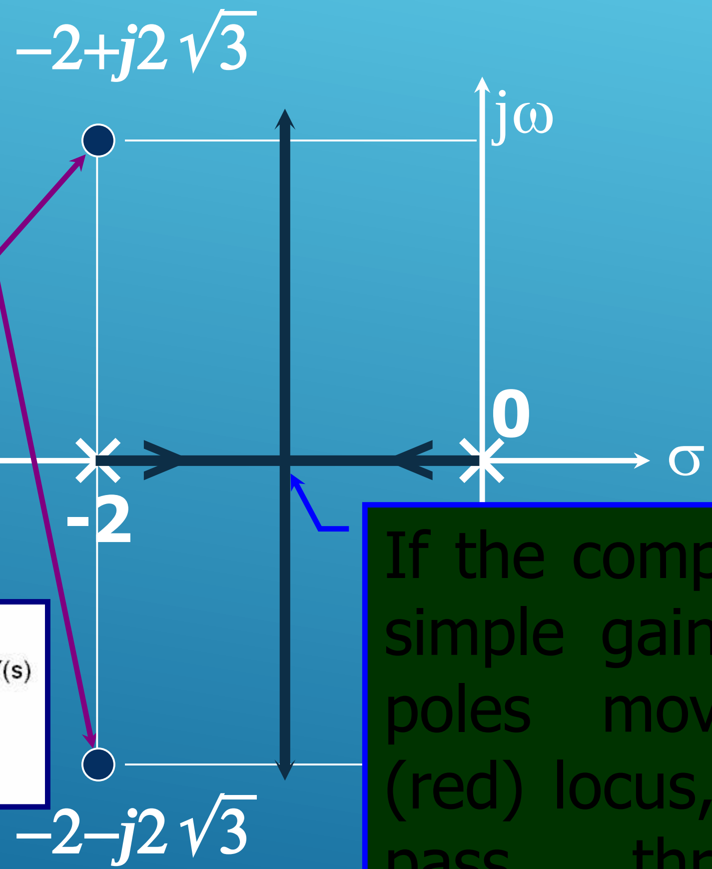


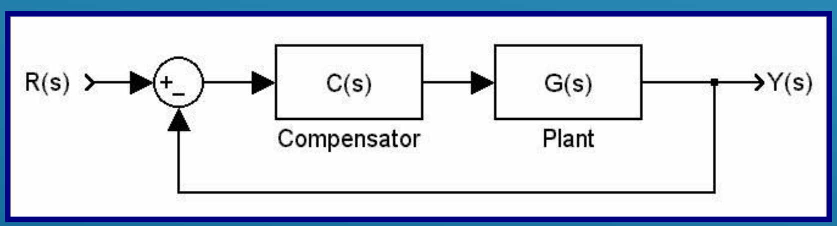
Calculate the angle deficiency (ϕ) at given locations, which are the desired CL pole locations, and then locate *p* and *z* to provide the $-\phi$ to satisfy angle condition. Then calculate K from the magnitude condition.

Design based on Root Locus
Lead Compensation - An Example

$$G(s) = \frac{4}{s(s+2)}$$

Desired CL poles





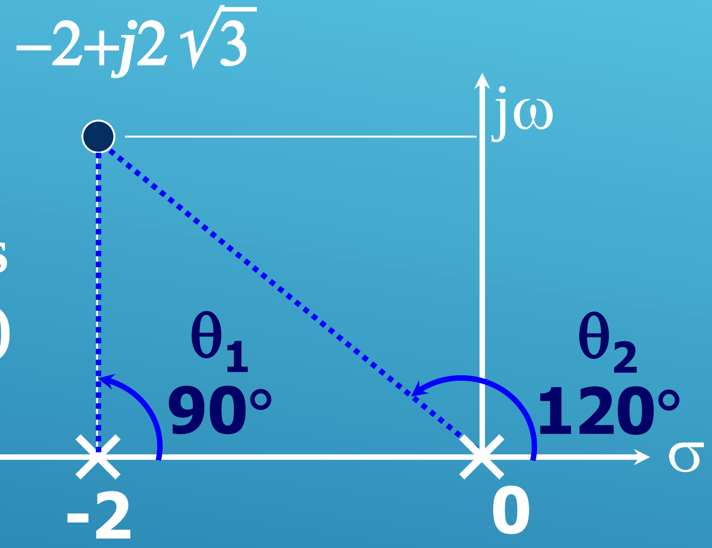
If the compensator is a simple gain, K, the CL poles move on this (red) locus, and do not pass through the desired locations...

Design based on Root Locus
Lead Compensation - An Example
Check Angle Condition

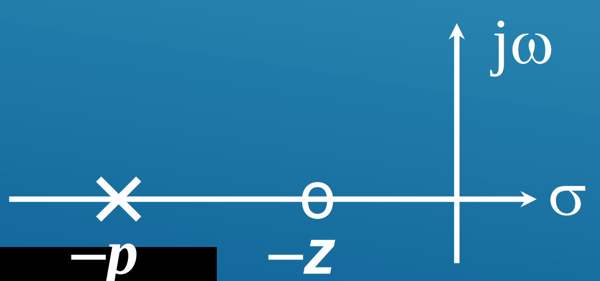
$$\Sigma\phi - \Sigma\theta = -210^\circ$$

From zeros
(No zeros!)

From poles
(Two poles)



Compensator must provide 30° to satisfy the angle condition. Remember the p & z configuration of the lead compensator.



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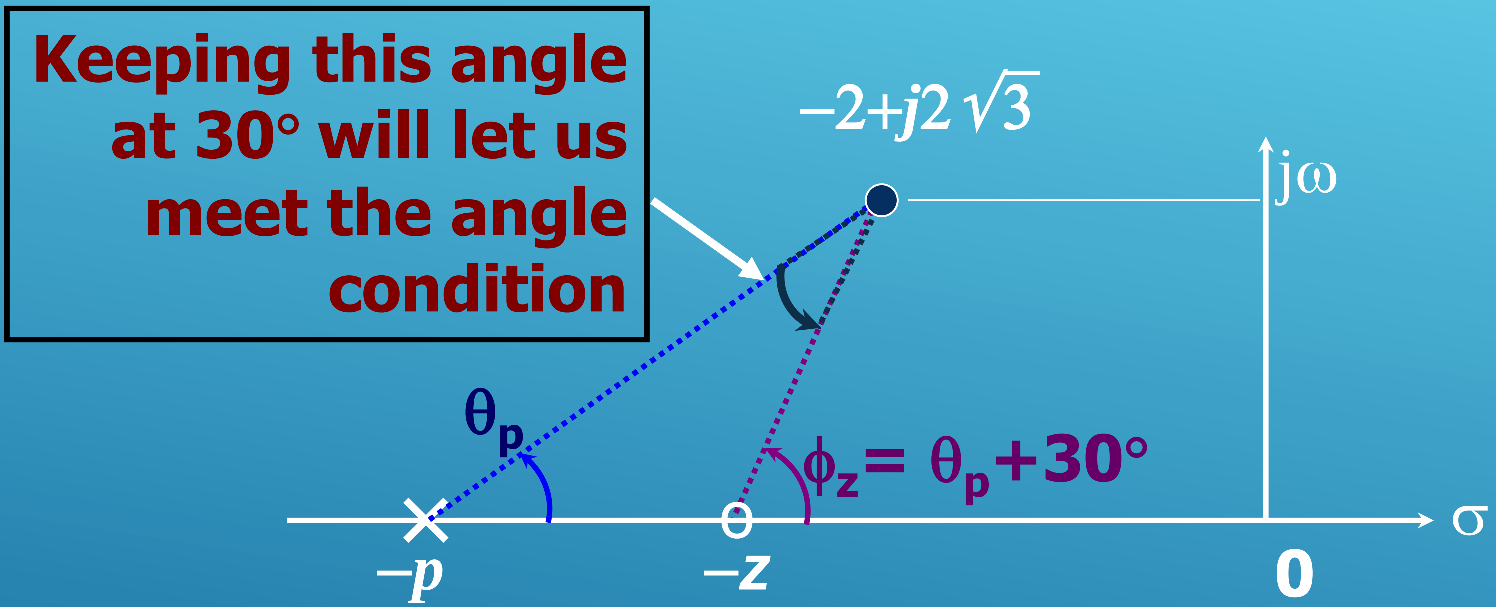
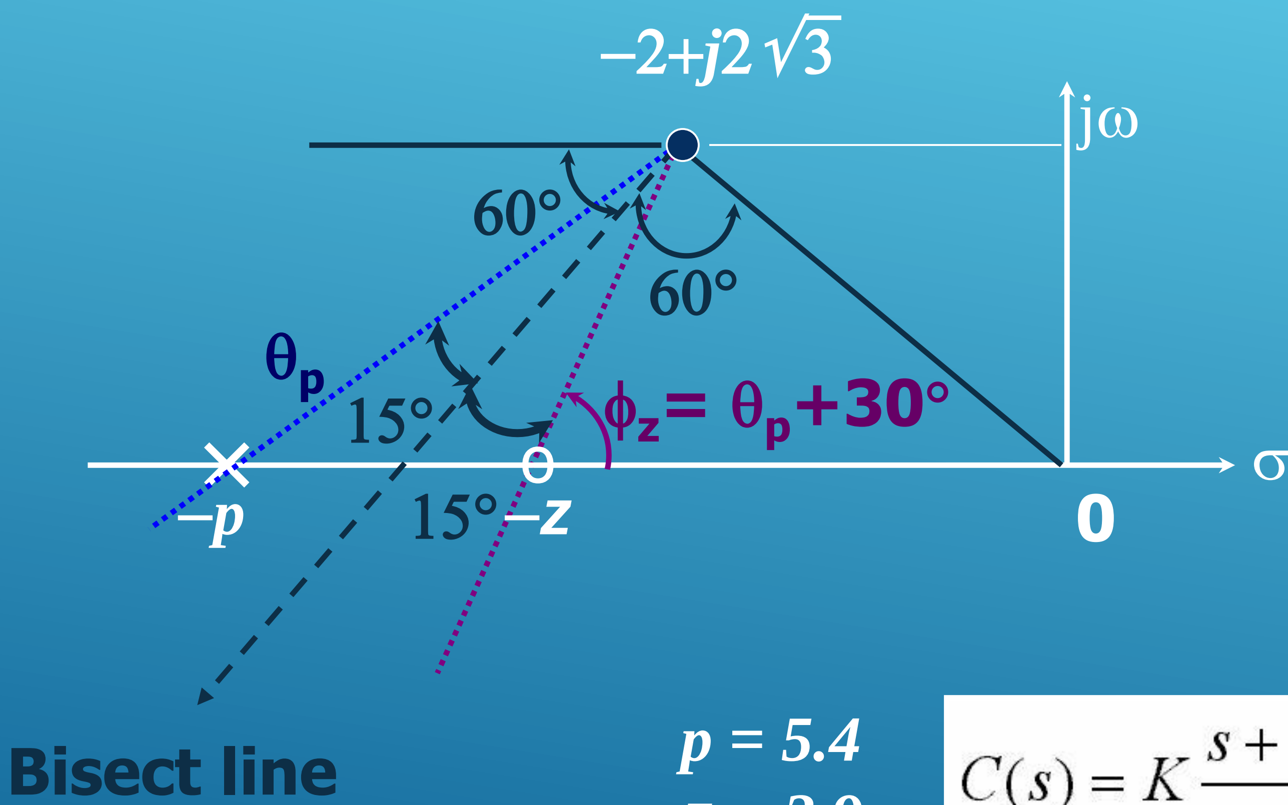
Design based on Root Locus
Lead Compensation - An Example



Clearly, there are lots of configurations providing 30° angle contribution? Which one should we choose?

Design based on Root Locus
Lead Compensation - An Example
To obtain the best K_v



$p = 5.4$
$z = 2.9$

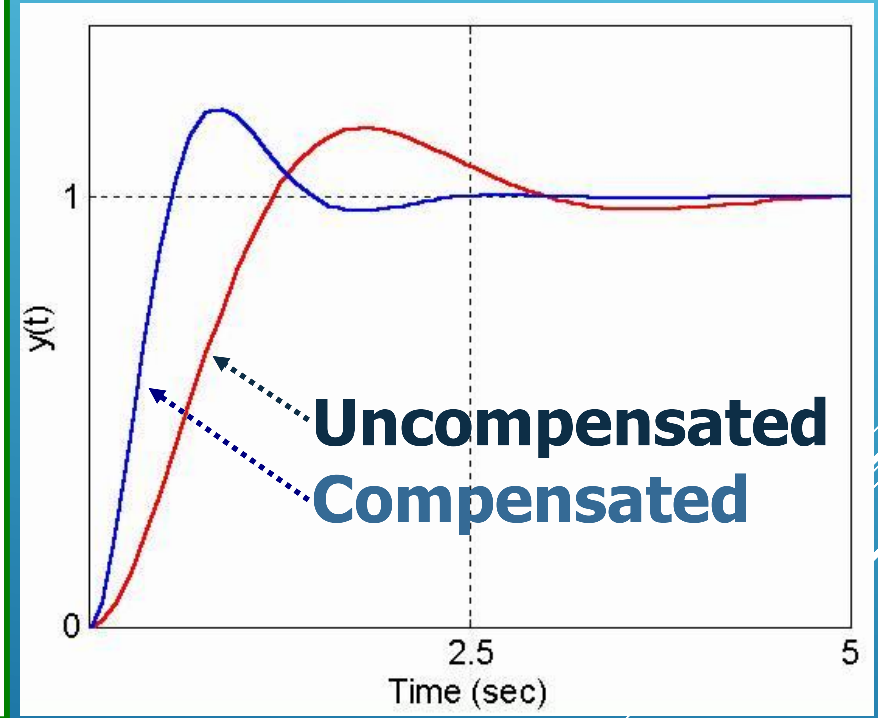
$$C(s) = K\frac{s+2.9}{s+5.4}$$

Design based on Root Locus
Lead Compensation - An Example
Determine K from the Magnitude Condition

$$\left|C(s)G(s)\right|_{s=-2\pm j2\sqrt{3}}=1$$

$$\left|K\frac{s+2.9}{s+5.4}\frac{4}{s(s+2)}\right|_{s=-2+j2\sqrt{3}}=1$$

$$K=4.7,\ C(s)=4.7\frac{s+2.9}{s+5.4}$$



Design based on Root Locus
Lead Compensation - An Example
Static Velocity Error Constant

Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞

$$
\begin{aligned}
K_v &= \lim_{s\to 0} sC(s)G(s) \\
&= \lim_{s\to 0} s\left(4.7\frac{s+2.9}{s+5.4}\right)\left(\frac{4}{s(s+2)}\right) \\
&= 5.02\ \sec^{-1}
\end{aligned}
$$

We obtained the maximum possible K_v as the bisect line is chosen

Design based on Root Locus
Lead Compensation
More general case: You are specified K_v

$$K_v = \lim_{s\to 0} sC(s)G(s) = \lim_{s\to 0} s\left(K\frac{s+z}{s+p}\right)\left(\frac{4}{s(s+2)}\right) = \frac{2Kz}{p}$$

$$\phi_z - \theta_p = 30^\circ = \arctan\left(\frac{2\sqrt{3}}{z-2}\right) - \arctan\left(\frac{2\sqrt{3}}{p-2}\right)$$

$$\left|K\frac{s+z}{s+p}\frac{4}{s(s+2)}\right|_{s=-2+j2\sqrt{3}} = 1 \Rightarrow K = 2\sqrt{3}\sqrt{\frac{(p-2)^2+12}{(z-2)^2+12}}$$

Angle Condition

Magnitude Condition

Solve the three equations for *z*, *p* and *K*

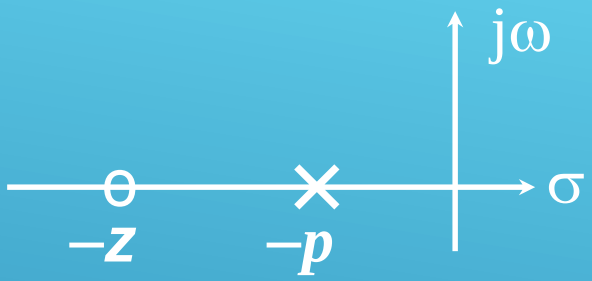
Design based on Root Locus
Lead Compensation - Remarks

- You have been given the CL poles explicitly in this example. In a more realistic problem, several specifications imply them. For example, the transient or steady state characteristics are described and you find out the required CL poles.

- Before jumping into equations, roughly sketch the root loci and make sure that you are on the right way.

Design based on Root Locus
Lag Compensation

$$C(s) = K\frac{s+z}{s+p}, \quad z > p$$



If the system performs well during transient period but poor during steady state, use a lag compensator to improve the steady state characteristics. Lag compensator increases the loop gain without modifying the locations of the dominant CL poles significantly. This is true as long as you locate *p* and *z* close to each other, furthermore, both are located close to origin.

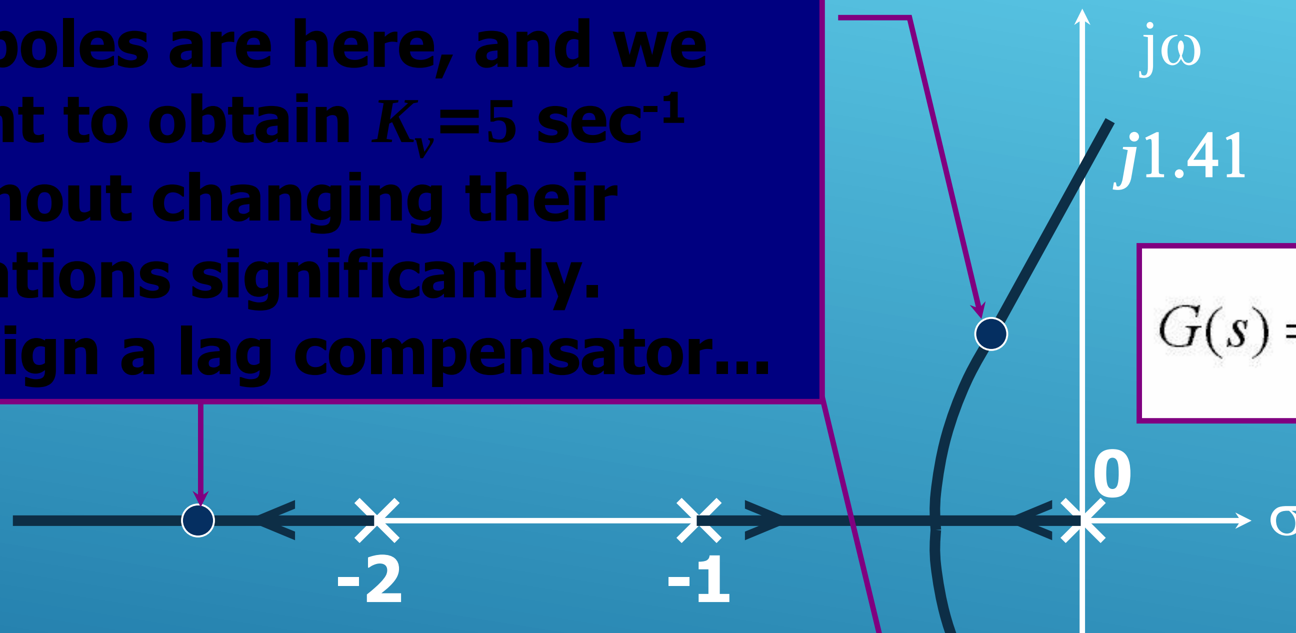
Design based on Root Locus
Lag Compensation

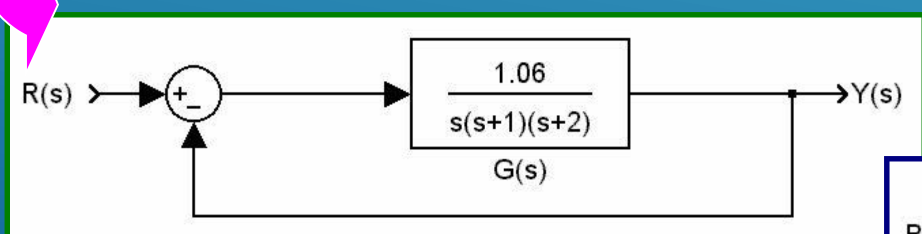
Typically, a desired static error constant is given. Since the angle contribution of the lag compensator is very small, the root loci does not change significantly. If this is not the case, i.e. if transient response is not satisfactory either, then you will be using a lag-lead compensator, which will be considered later…

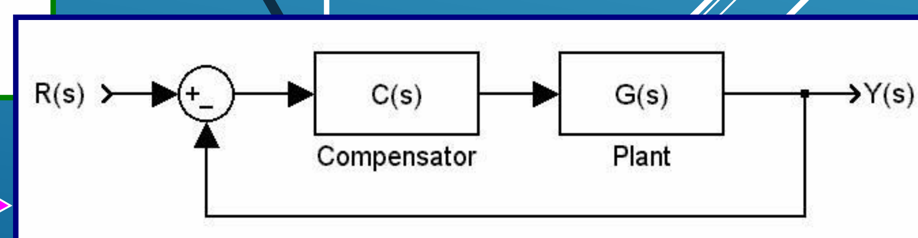
Design based on Root Locus
Lag Compensation - An Example

CL poles are here, and we want to obtain K_v=5 sec^{-1} without changing their locations significantly. Design a lag compensator…

$$G(s) = \frac{1.06}{s(s+1)(s+2)}$$

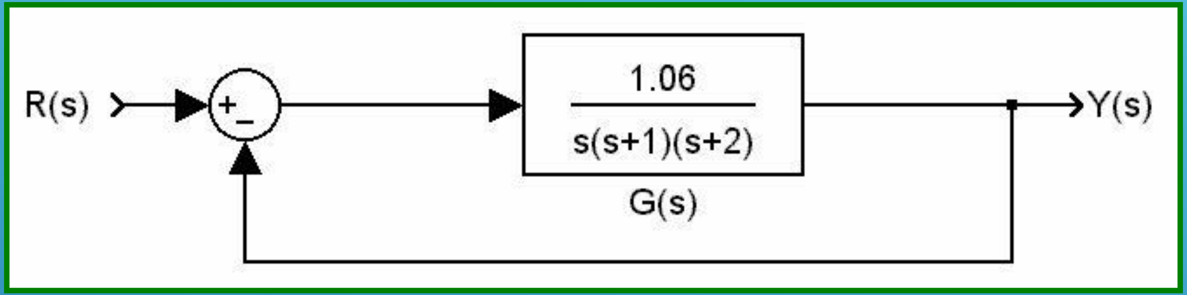






Design based on Root Locus
Lag Compensation - An Example

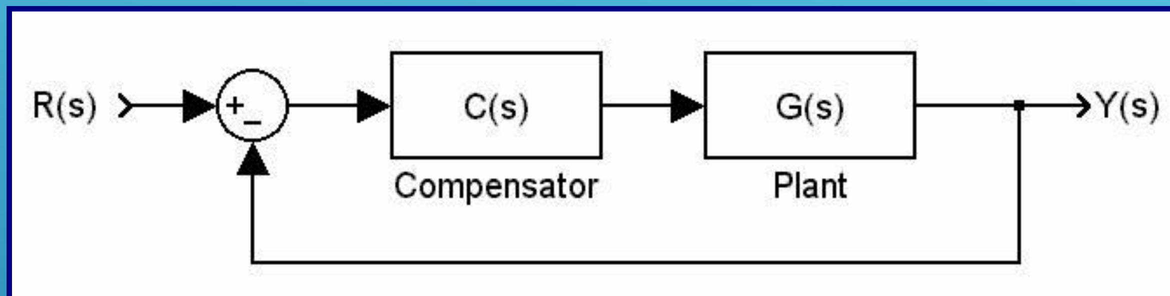
With this configuration,



- **The dominant CL poles are at $s = -0.3307 \pm j0.5864$**
- **The damping ratio is $\zeta = 0.491$**
- **The static velocity error constant is $K_v = 0.53$ sec^{-1}**
- **We want $K_v = 5$ sec^{-1}**

Design based on Root Locus
Lag Compensation - An Example

Adopt this configuration,



Locate *z* and *p* very close to origin

$$C(s) = K\frac{s+z}{s+p}, \quad z > p$$

Design based on Root Locus
Lag Compensation - An Example

$$G(s) = \frac{1.06}{s(s+1)(s+2)} \qquad K_v = \lim_{s \to 0} sG(s) = 0.53$$

$$C(s)G(s) = K\frac{s+z}{s+p}\frac{1.06}{s(s+1)(s+2)}$$

$$K_{v\mathrm{NEW}} = \lim_{s \to 0} sC(s)G(s) = K\frac{z}{p}0.53$$

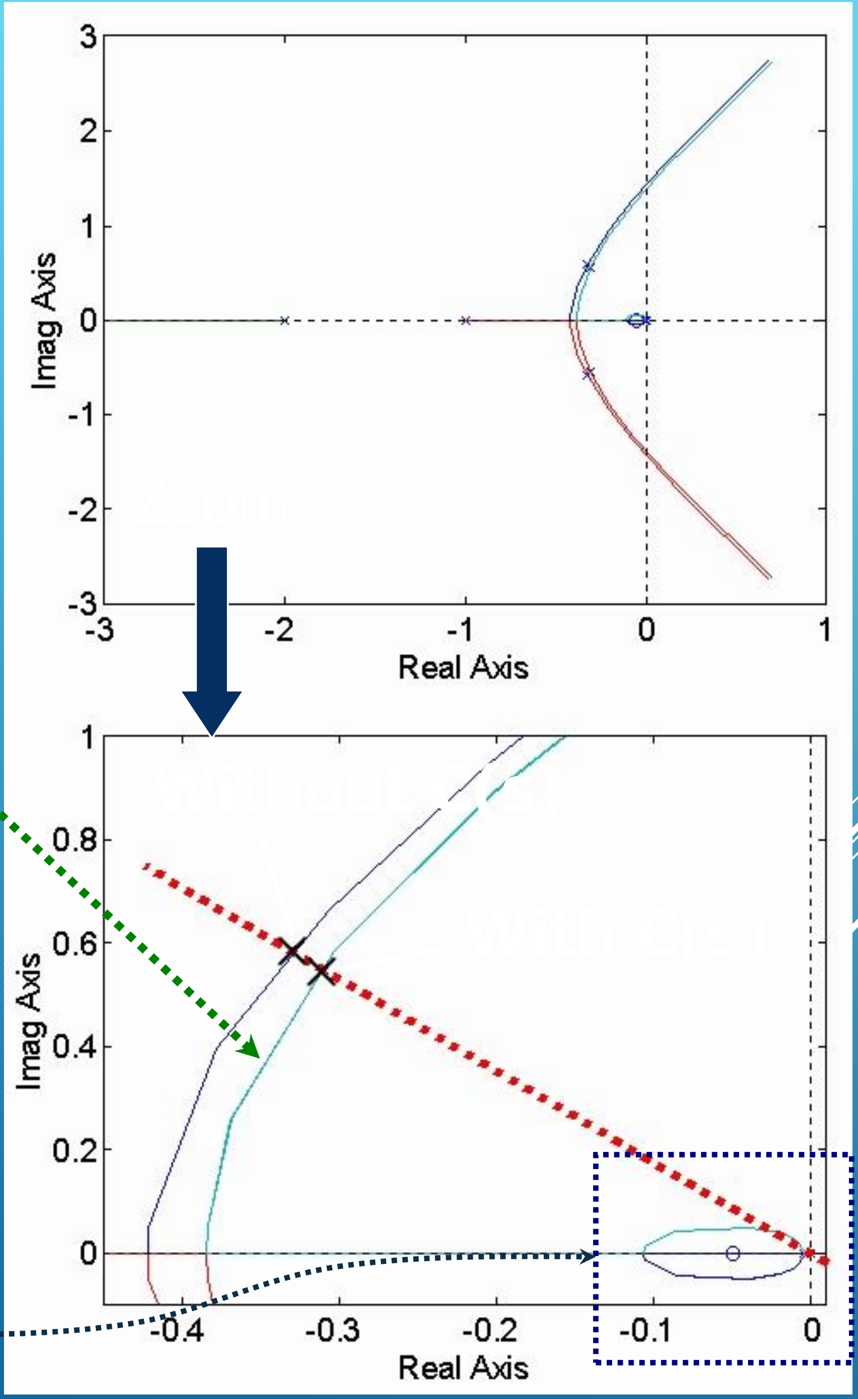
- $K_{vNEW}/K_v \cong 10$, so set z=0.05 and p=0.005
- Calculate angle contribution, which is ≈4°
- This will slightly change the root locus
- Tune K to keep ζ same (ζ=0.491), K=1.0235

Design, R-Locus
Lag Comp. Example

- What would happen if there were no K adjustment?

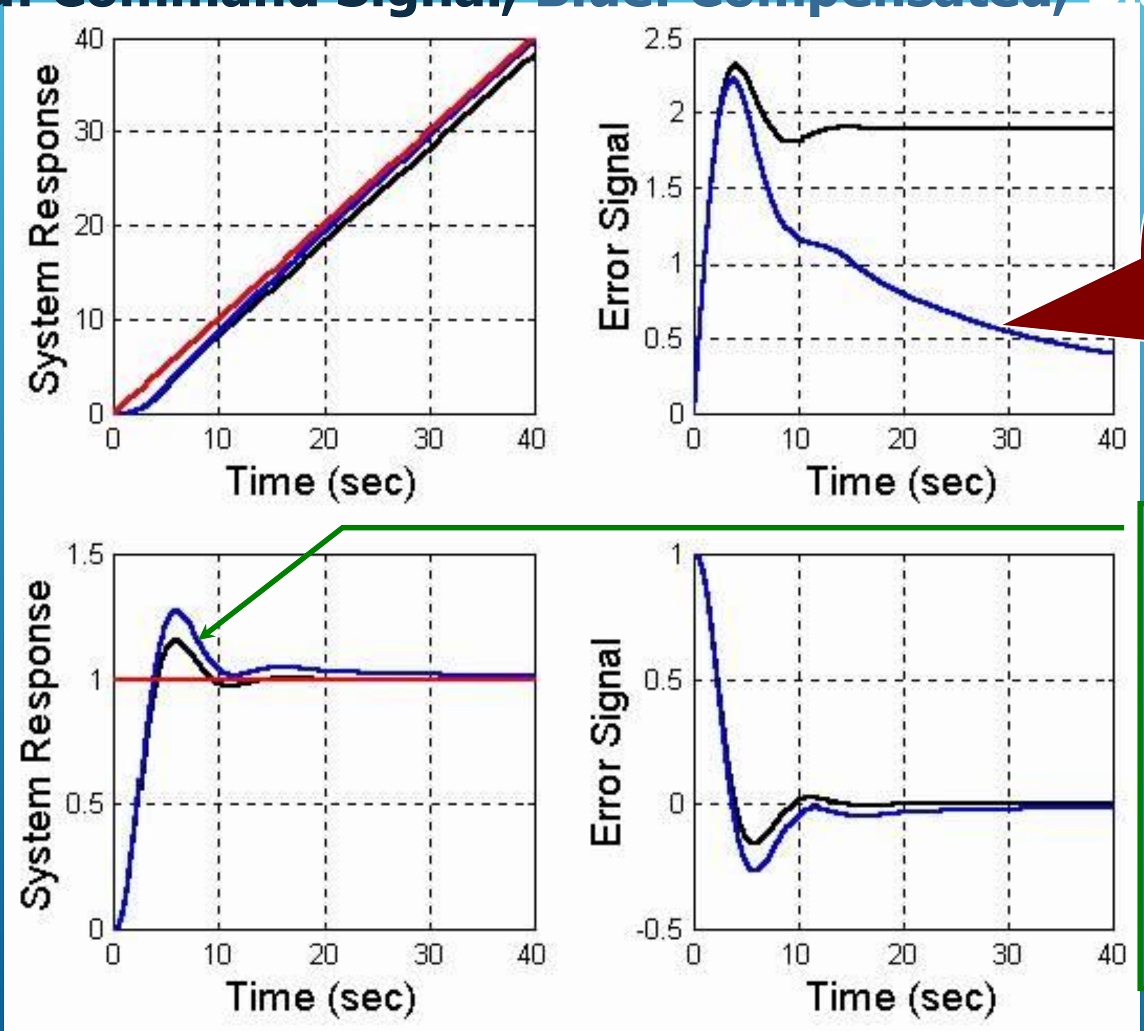
The answer is on the graph. Here you see two loci, which are almost identical. Nevertheless, you have to find the correct value of K…

Pay attention, the pole and the zero of C(s) are here



Design based on Root Locus
Lag Compensation - An Example

Red: Command Signal, Blue: Compensated, Black: Uncompensated



This was what we aimed. Curve goes to 0.1

Speed of the response has decreased because of the lag compensation

Design based on Root Locus
Lag-Lead Compensation

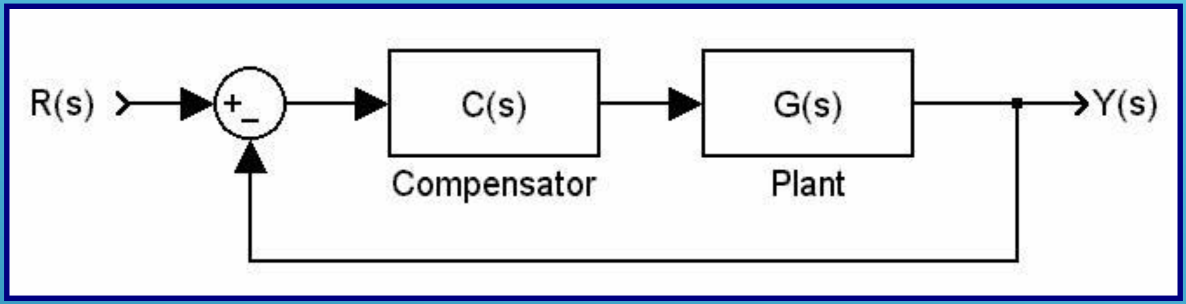
- **Lead compensation speeds up the response and increases the stability of the system.**
- **Lag compensation improves the steady state accuracy but reduces the speed of the response.**

- **If the design specifications require both a fast response and better steady state characteristics, a Lag-Lead compensator is used.**

Design based on Root Locus
Lag-Lead Compensation

- Calculate the relevant variables (ω_n, ζ, ω_d etc)
- Firstly, design the Lead Compensator
  - Calculate the angle deficiency
  - Locate the zero of the compensator
  - Locate the pole such that the angle condition is met
- Secondly, design the Lag Compensator
  - Locate its pole close to zero
  - According to steady state response specs., locate the zero
  - Check the angle contribution of Lag Comp.
  - If necessary, retune the gain so that ζ is kept at its desired value.

Design based on Root Locus
Lag-Lead Compensation - An Example



$$G(s) = \frac{4}{s(s+0.5)}$$

Design Specifications

- Dominant CL poles are desired to have $\zeta = 0.5$
- Desired Undamped natural frequency is $\omega_n = 5$ rad/sec
- Desired Static velocity error constant is $K_v = 80\ \text{sec}^{-1}$

Design based on Root Locus
Lag-Lead Compensation - An Example
Step 1: Calculate the relevant variables
If there is no compensator, you have

$$G(s) = \frac{4}{s(s+0.5)}$$

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{4}{s^2 + 0.5s + 4}$$

$$\zeta = 0.125 \text{ and } \omega_n = 2 \text{ rad/sec}$$

$$K_v = \lim_{s \to 0} sG(s) = 8 \text{ sec}^{-1}$$

Desired
ζ = 0.5

Desired
K_v = 80 sec^{-1}

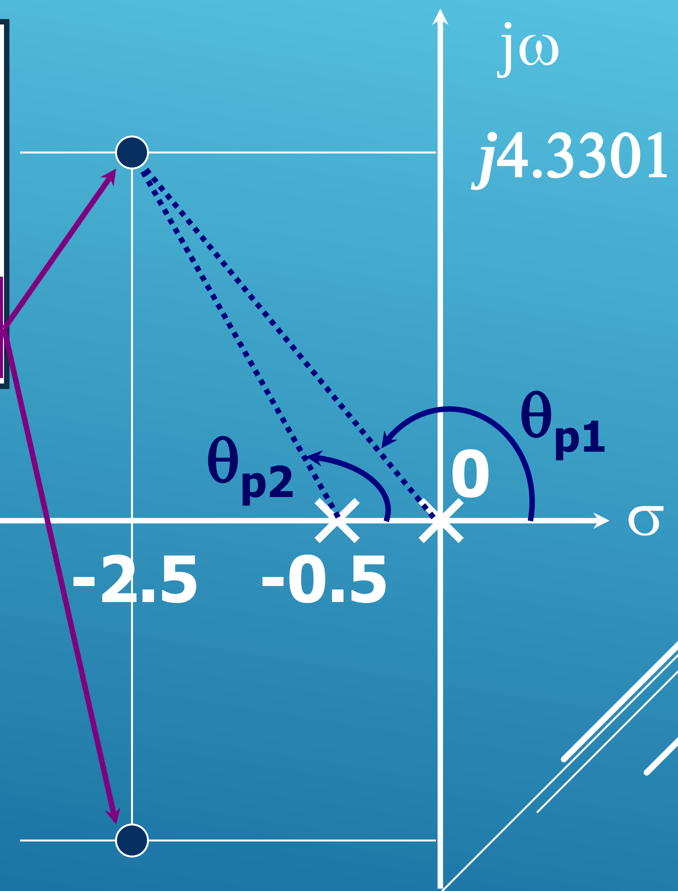
Desired
ω_n = 5 rad/sec

Design based on Root Locus
Lag-Lead Compensation - An Example
Step 2: Design the Lead Compensator

$$-\zeta\omega_n = -0.5 * 5 = -2.5$$

$$j\omega_d = j\sqrt{1-\zeta^2}\,\omega_n = j4.3301$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d = \boxed{-2.5 \pm j4.3301}$$



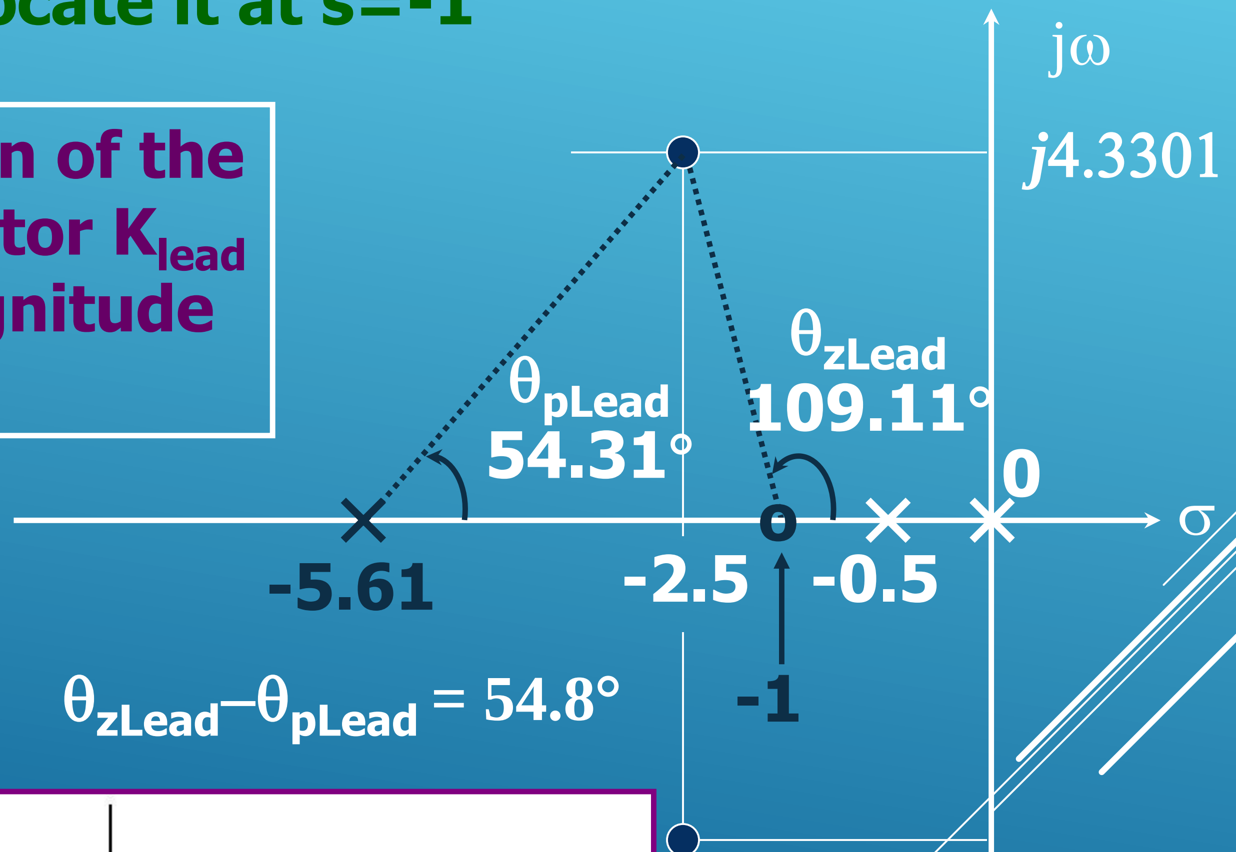
Now Calculate the angle deficiency

$\Sigma\phi - \Sigma\theta = -234.8°$

Lead Controller will contribute $54.8°$ **to make sure that** $\Sigma\phi - \Sigma\theta = \pm(2k+1)180°$

Design based on Root Locus
Lag-Lead Compensation - An Example
Step 2: Locate the zero of Lead Comp.
Let's locate it at s=-1

Now set the gain of the Lead Compensator K_{lead} i.e. refer to magnitude condition



$$\left| K_{lead} \frac{s+1}{s+5.61} \frac{4}{s(s+0.5)} \right| = 1 \Rightarrow K_{lead} = 6.94$$

Design based on Root Locus
Lag-Lead Compensation - An Example
Step 3: Design the Lag Compensator

$$C_{lead}(s)G(s) = \left(6.94\frac{s+1}{s+5.61}\right)\left(\frac{4}{s(s+0.5)}\right)$$

$$K_{v(new)} = \lim_{s\to 0} sC_{lead}(s)G(s) = 9.9\ \text{sec}^{-1}$$

$$\frac{K_{v(new)}}{K_{v(desired)}} = \frac{9.9\ \text{sec}^{-1}}{80\ \text{sec}^{-1}} = 0.124$$

When s=0, the Lag compensator must increase the loop gain by 1/0.124 ≈ 8.06

Design based on Root Locus
Lag-Lead Compensation - An Example

Step 3: Locate the zero of Lag Compensator
Let's locate it at s=-0.1

$$C_{lag}(s) = K_{lag}\frac{s+0.1}{s+0.0124}$$

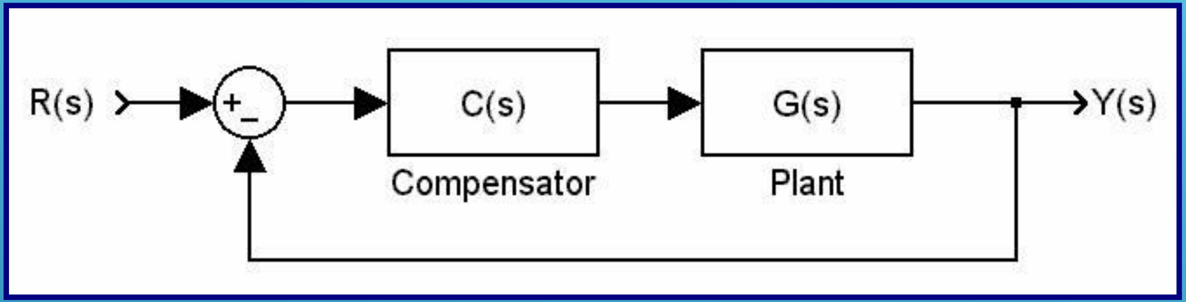
$$C_{lead}(s)C_{lag}(s)G(s) = \left(6.94\frac{s+1}{s+5.61}\right)\left(K_{lag}\frac{s+0.1}{s+0.0124}\right)\left(\frac{4}{s(s+0.5)}\right)$$

$$K_v = \lim_{s\to 0} sC_{lead}(s)C_{lag}(s)G(s) = K_{lag}79.8114\ \sec^{-1}$$

$$\text{Angle Contribution is} : 0.8791^{\circ}$$

Angle contribution is acceptably small. However, this has slightly changed z. A very tiny tuning can be made if the design specifications are too stringent. For this example, there is no need to do so, keep K_{lag}=1.

Design based on Root Locus
Lag-Lead Compensation - An Example

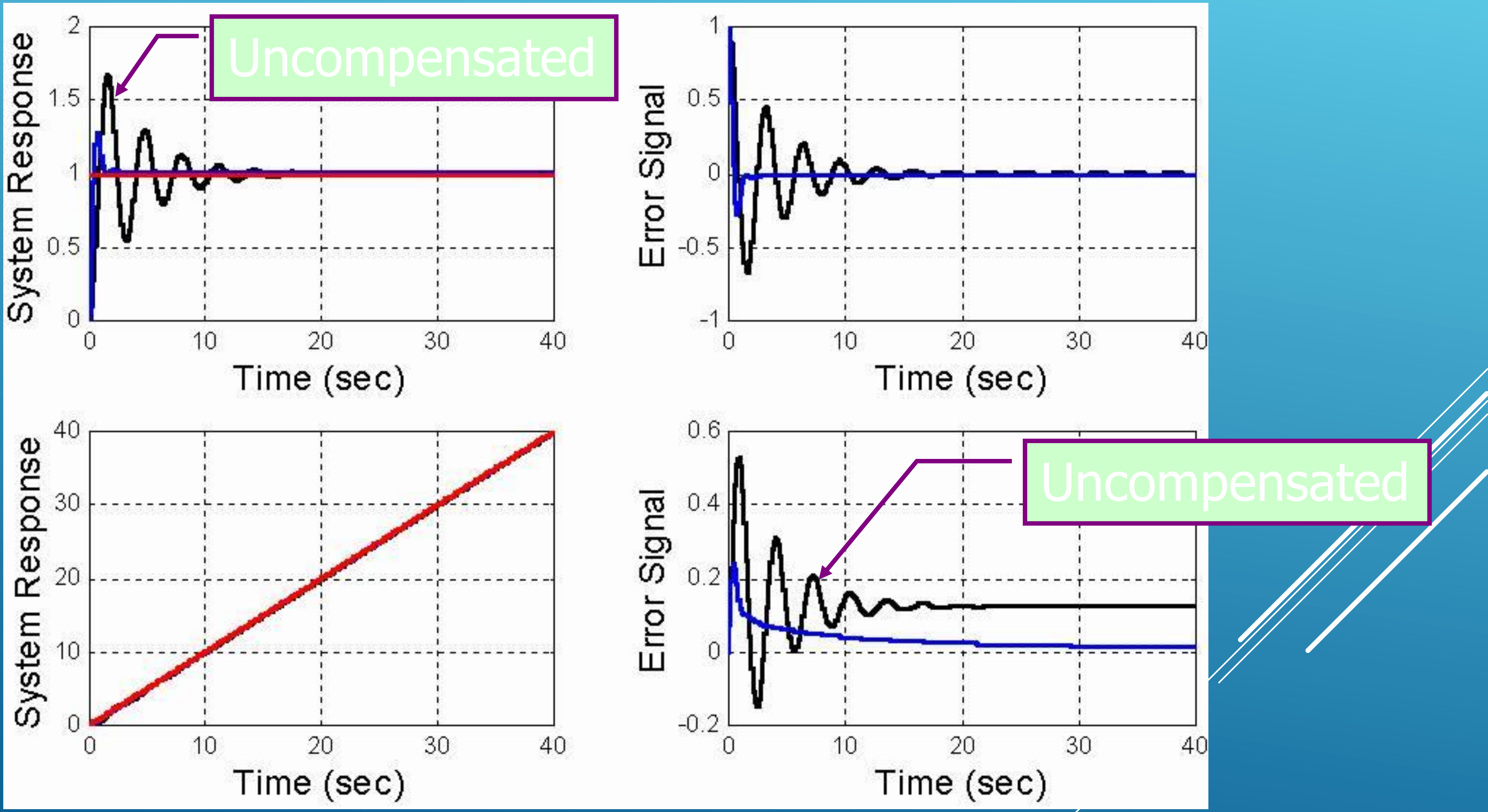


$$G(s) = \frac{4}{s(s+0.5)}$$

$$C(s) = 6.94\frac{(s+1)(s+0.1)}{(s+5.61)(s+0.0124)}$$

Now, test and see whether the design specifications are met or not...

Design based on Root Locus
Lag-Lead Compensation - An Example
Step and Ramp Responses



Design based on Root Locus
Lag-Lead Compensation - An Example
A Comparison

Simple K_{lag}
Good enough

$K_{lag} = 1$
$K_v = 79.81139669944224\ sec^{-1}$
$\zeta = 0.49452458450471$
CL Poles: $s=-2.44946613086810 \pm j4.30511842727874$
$s=-1.12268288809756$ and $s=-0.10078485016624$

K_v is exact

$K_{lag} = 80/79.81139669944224=1.00236311239193$
$Kv = 80\ sec^{-1}$
$\zeta = 0.49388974530242$
CL Poles: $s=-2.44966485404744 \pm j4.31279190736033$
$s=-1.12228732098688$ and $s=-0.10078297091824$

ζ is exact

$K_{lag} = 0.97999709075950$
$Kv = 78.21493657490576\ sec^{-1}$
$\zeta = 0.50000000000001$
CL Poles: $s=-2.44773023820451 \pm j4.23959313579281$
$s=-1.12613839738740$ and $s=-0.10080112620358$

Remarks on Root Locus and Design Based on Root Locus

- Manipulating the roots and the poles of the closed loop system may yield the desired solution, which can be sought by root locus method.

- Stringent design specs. carry priority. Meeting them precisely may require computer based analysis and design.

- It is useful to know the following Matlab functions: rlocus(.,.), rlocfind(.,.) and rltool. The last one lets you play with the poles and zeros to see their effects on responses and several other control engineering design tools.

Frequency Response Analysis
Bode Plots - First Order Factors
How to do with Matlab?

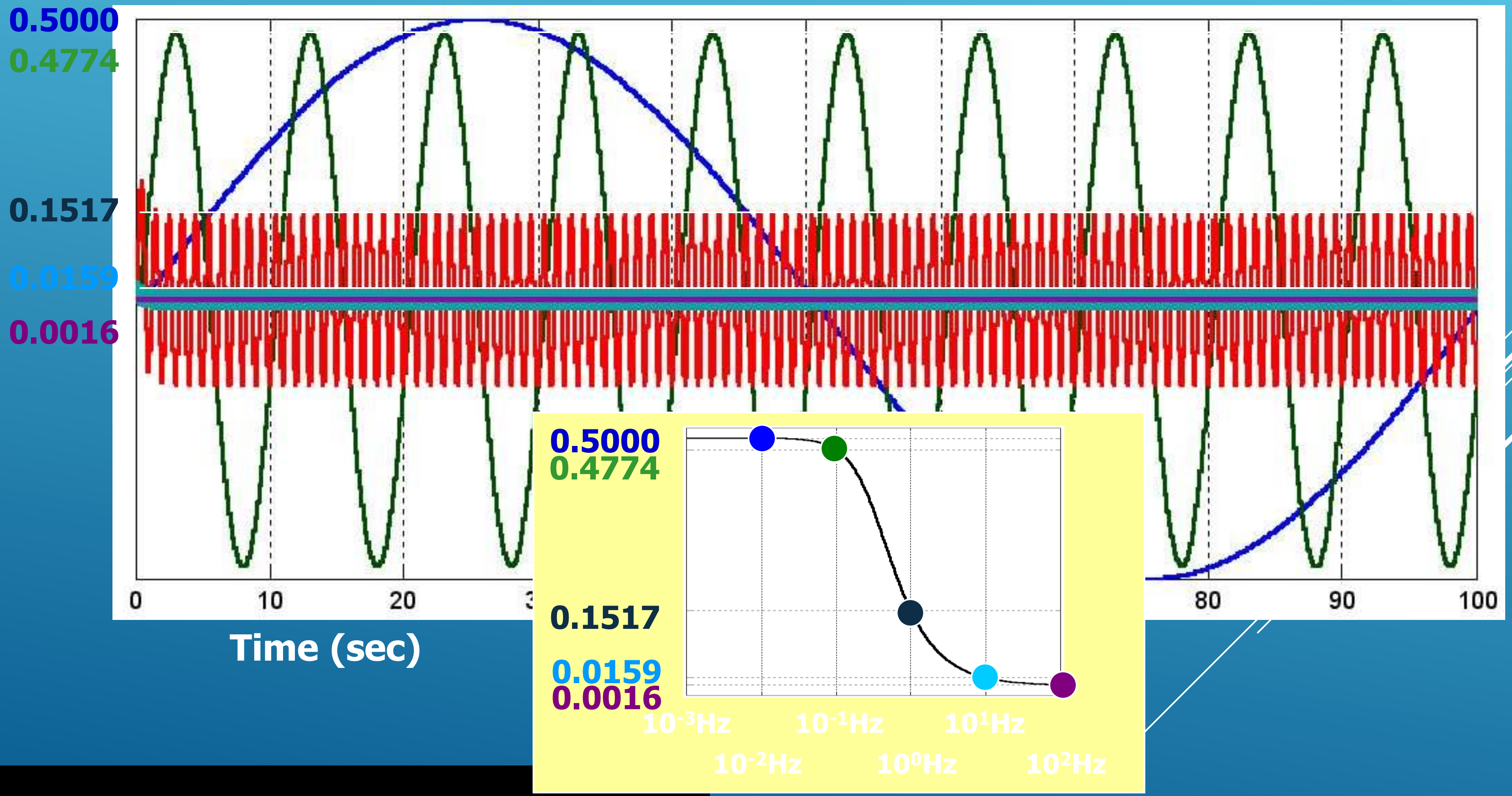
$$G(s) = \frac{3}{s+2}$$

```
» numerator = [3];
» denominator = [1 2];
» w=logspace(-2,2,100); % 100 points btw. 10^-2 and 10^2
» bode(numerator,denominator,w)
```

Frequency Response Analysis
Bode Plots - First Order Factors

$$G(s) = \frac{1}{s+2}$$

Input is sin(2πft) for f=0.01 Hz
0.1 Hz, 1 Hz, 10 Hz, 100 Hz

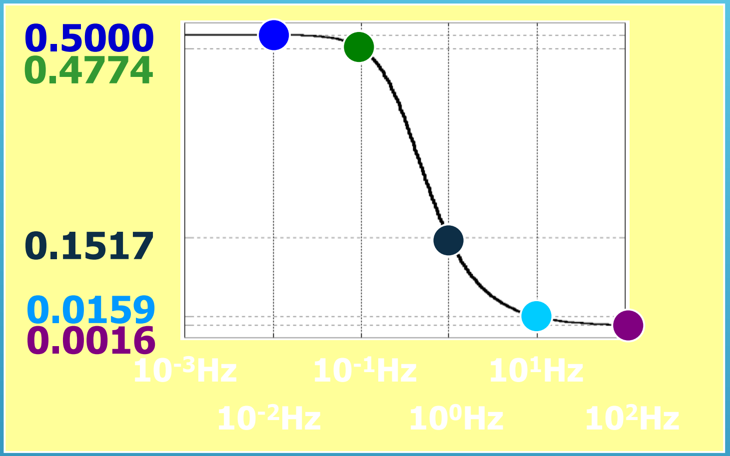


[redacted]

Frequency Response Analysis
Bode Plots - First Order Factors
Some Matlab Work

$$G(s) = \frac{1}{s+2}$$

```
» num = 1;
» den = [1  2];
» w = 2*pi*[0.01  0.1  1  10  100];
» [Magnitude,Phase]=bode(num,den,w);
» Magnitude'
```



ans =

0.4998	0.4770	0.1517	0.0159	0.0016
0.5000	**0.4774**	**0.1517**	**0.0159**	**0.0016**

As the input frequency increases, the amplitude of the sinusoidal signal at the output decreases.

Frequency Response Analysis
Bode Plots - First Order Factors - An Example

$$G(s)=\frac{16(s+2)(s+10)}{s(s+40)(s+100)}=K\frac{(s/2+1)(s/10+1)}{s(s/40+1)(s/100+1)}$$

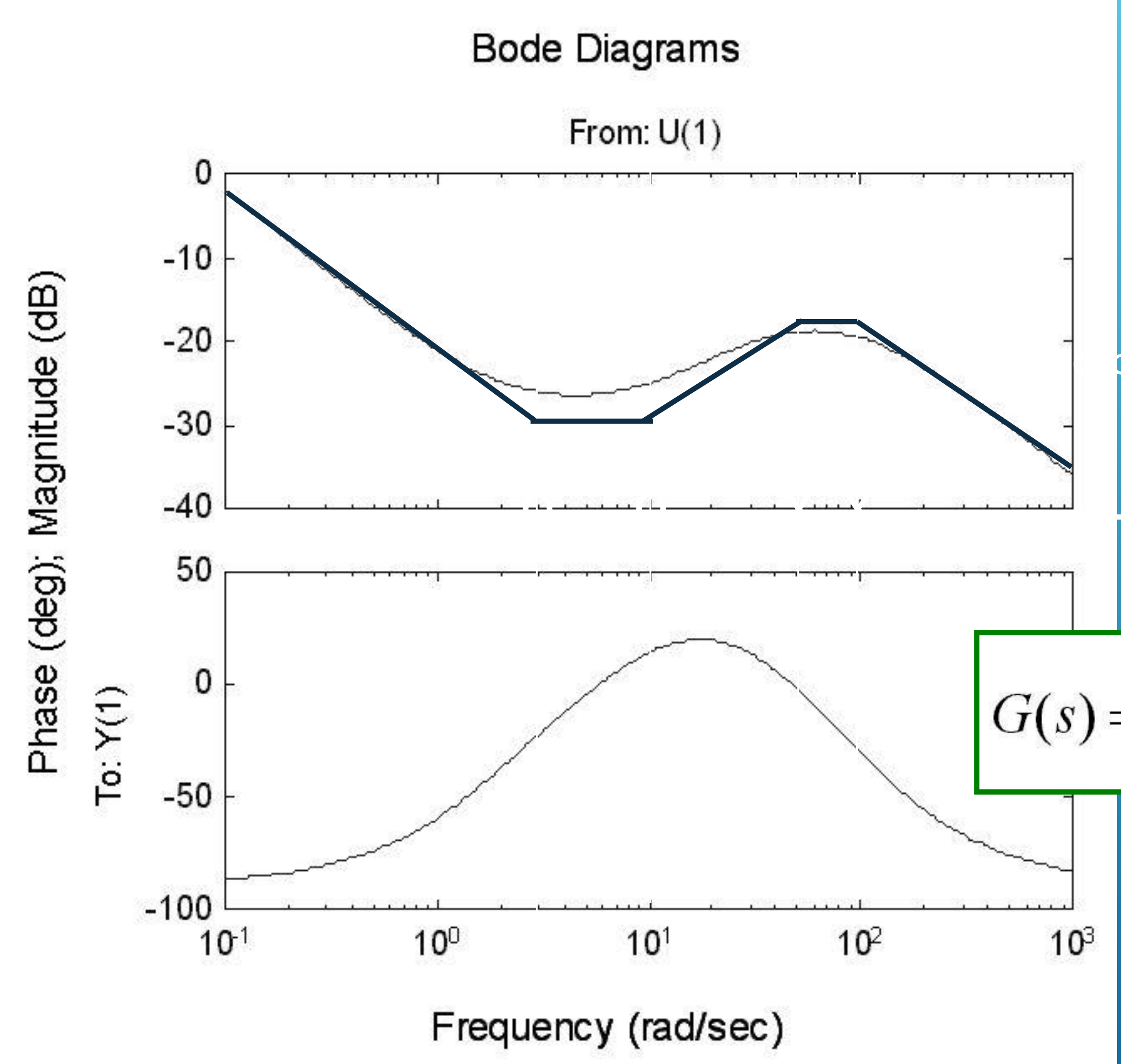
$$G(s)=0.08\frac{(1+0.5s)(1+0.1s)}{s(1+0.025s)(1+0.01s)}$$

$$G(j\omega)=0.08\frac{(1+j0.5\omega)(1+j0.1\omega)}{j\omega(1+j0.025\omega)(1+j0.01\omega)}$$

$$|G(j\omega)|=0.08\frac{|1+j0.5\omega||1+j0.1\omega|}{|j\omega||1+j0.025\omega||1+j0.01\omega|}$$

$$\angle G(j\omega)=\arctan(0.5\omega)+\arctan(0.1\omega)-90^{\circ}$$
$$-\arctan(0.025\omega)-\arctan(0.01\omega)$$

Frequency Response Analysis
Bode Plots - First Order Factors - An Example



Normally, we do not mark these poles and zeros!

$$G(s)=\frac{16(s+2)(s+10)}{s(s+40)(s+100)}$$

Frequency Response Analysis
Bode Plots - First Order Factors

- Set a starting frequency (w_0), and calculate |G(jw)| at that frequency.
- Then Sweep the frequency axis. If G(s) has n poles (zeros) at zero, start with a curve of slope -20n (20n) dB/decade.
- Continue sweeping: At every pole (zero) decrease (increase) the slope 20m dB/decade, where m is the multiplicity of that pole (zero).

Frequency Response Analysis
Bode Plots - Quadratic Factors

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + 2\zeta\left(j\frac{\omega}{\omega_n}\right)}$$

$$20\log\left|G(j\omega)\right| = -20\log\left(\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right)$$

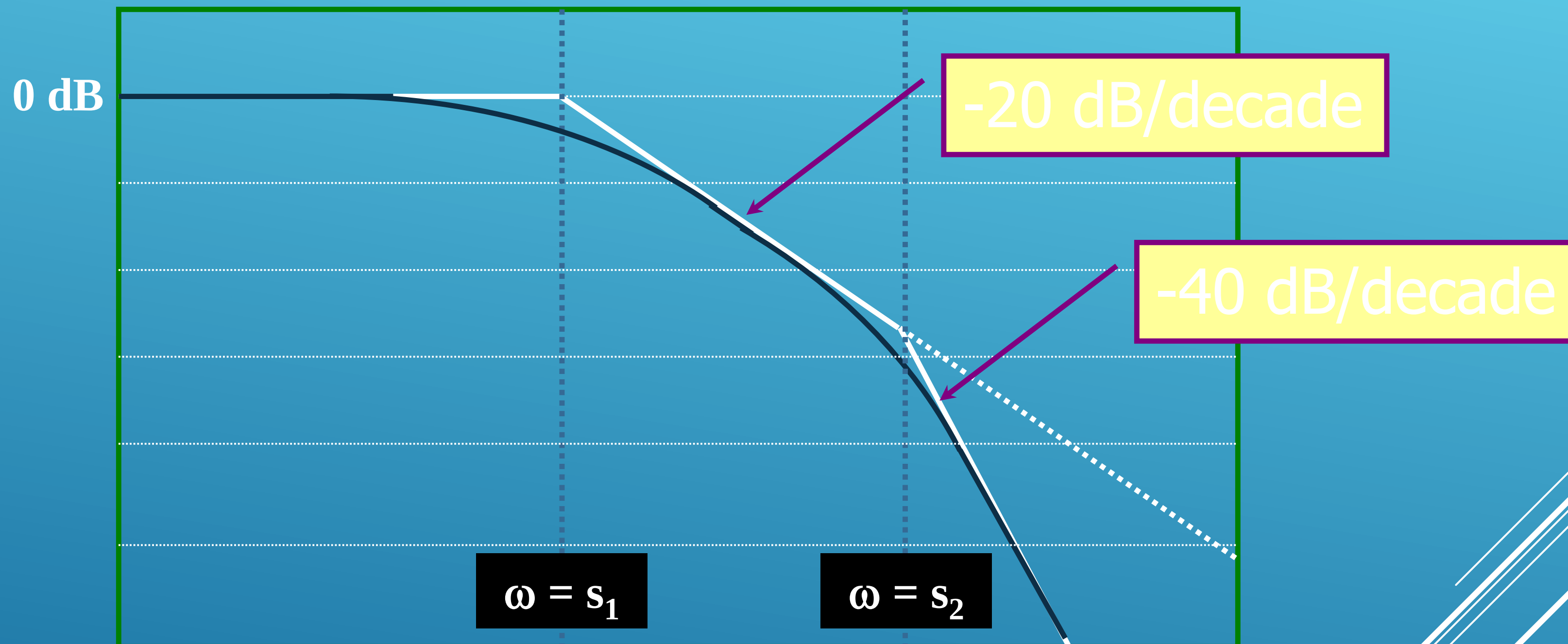
$$\angle G(j\omega) = -\arctan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

[redacted]

Frequency Response Analysis
Bode Plots - Quadratic Factors

- $\zeta>1$ There are two real poles
- $\zeta=1$ There two real poles at $s=-w_n$
- $\zeta=0$ Poles are on the imaginary axis
- $0<\zeta<1$ Several situations… We will see

Frequency Response Analysis
Bode Plots - Quadratic Factors
ζ>1: You have two real poles



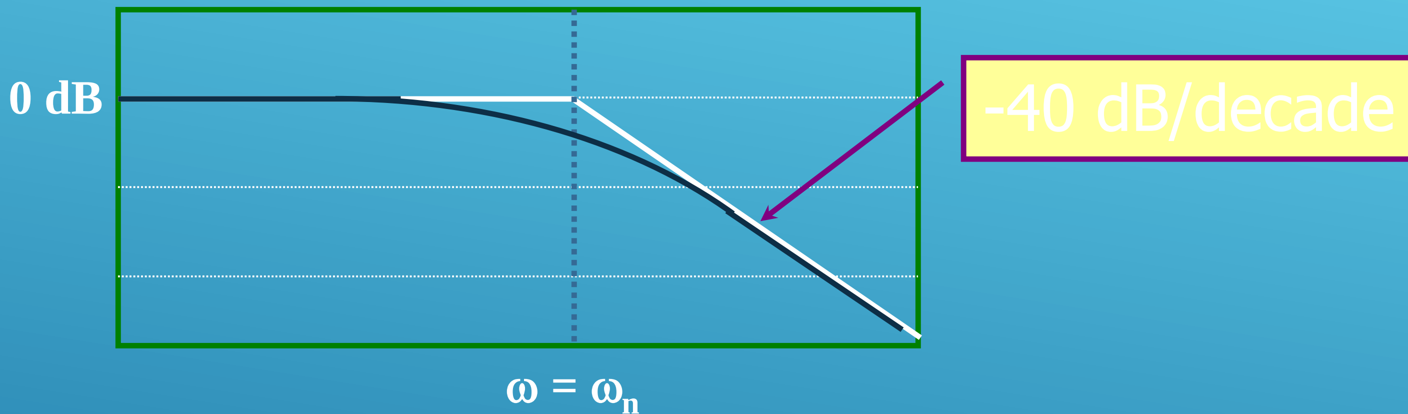
$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{\omega_n^2}{(s-s_1)(s-s_2)}$$

$$s_{1,2}=-\omega_n\left(\zeta\pm\sqrt{\zeta^2-1}\right)$$

[redacted]

Frequency Response Analysis
Bode Plots - Quadratic Factors

ζ=1: You have two real poles at s=$-\omega_n$

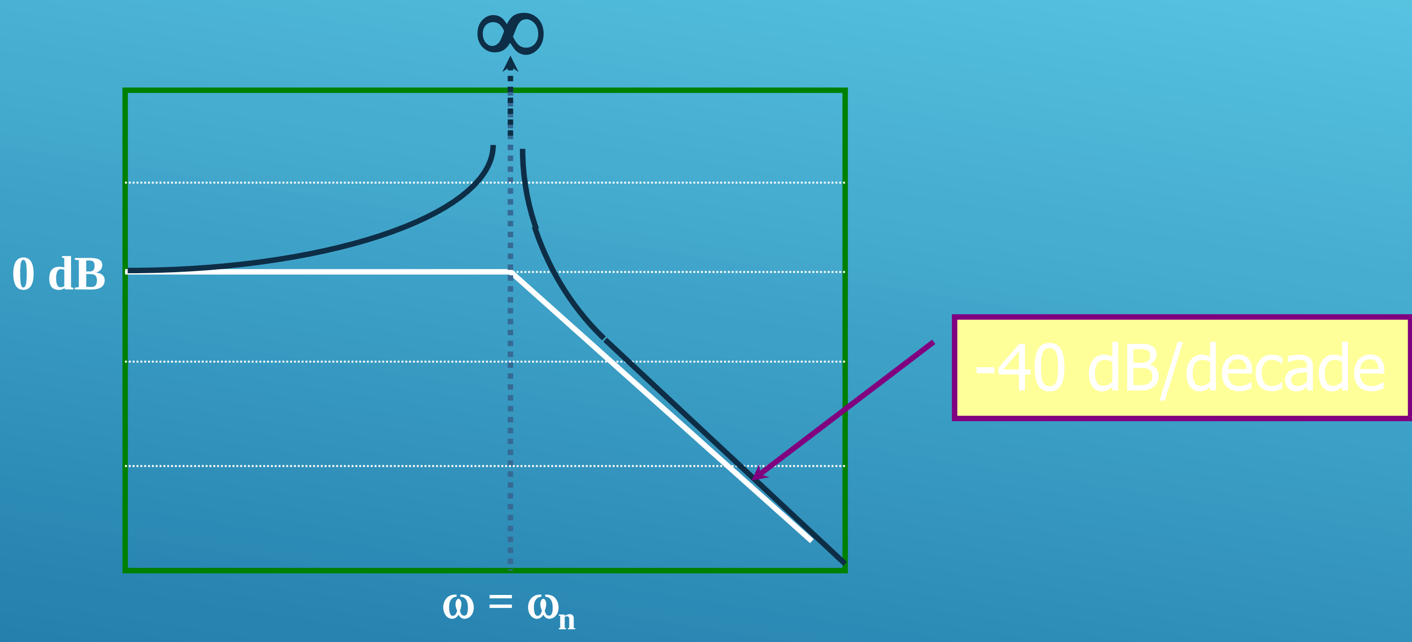


$$G(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{\left(s + \omega_n\right)^2}$$

$$G(j\omega) = \frac{1}{\left(j\dfrac{\omega}{\omega_n} + 1\right)^2}$$

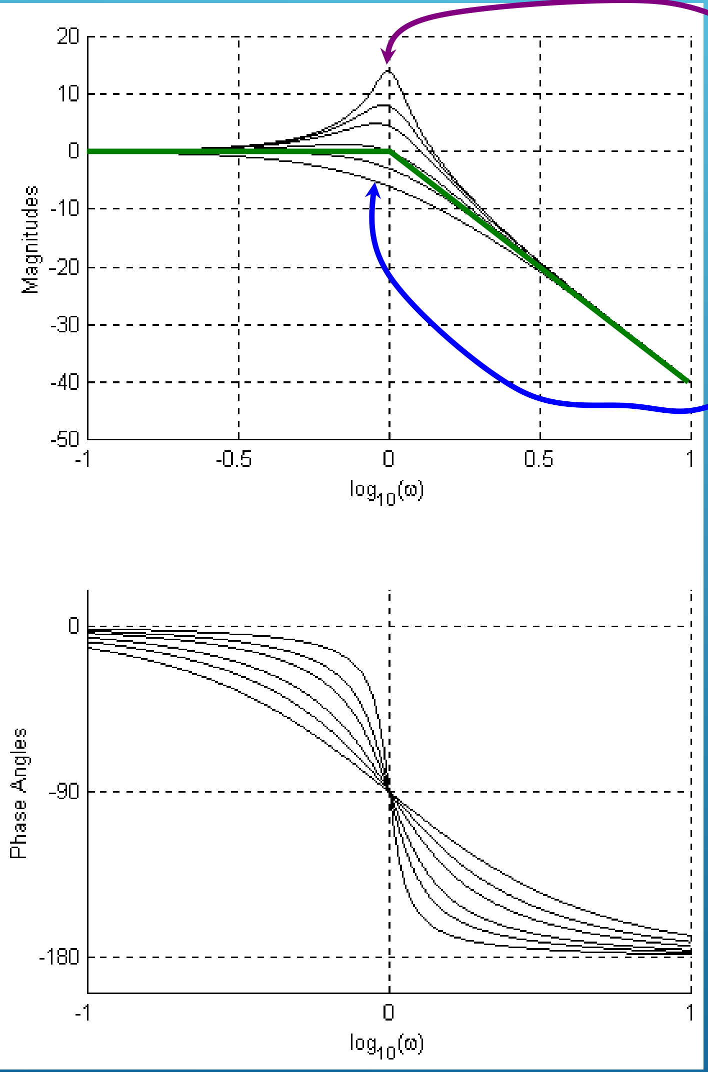
[redacted]

Frequency Response Analysis
Bode Plots - Quadratic Factors
ζ=0: Poles are on the imaginary axis



$$G(s)=\frac{\omega_n^2}{s^2+\omega_n^2} \qquad |G(j\omega)|=\frac{1}{\left|1-\left(\frac{\omega}{\omega_n}\right)^2\right|}$$

Frequency Response Analysis
Bode Plots - Quadratic Factors - 0<ζ<1



ζ=0.1
ζ=0.2
ζ=0.3
ζ=0.5
ζ=0.7
ζ=1.0

Resonant Frequency (ω_r)

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \quad \text{for} \quad 0 \le \zeta \le 1/\sqrt{2}$$

If $\zeta > 1/\sqrt{2}$ There is no resonant peak

Frequency Response Analysis
Bode Plots
Minimum-Phase Systems and Nonminimum-Phase Systems

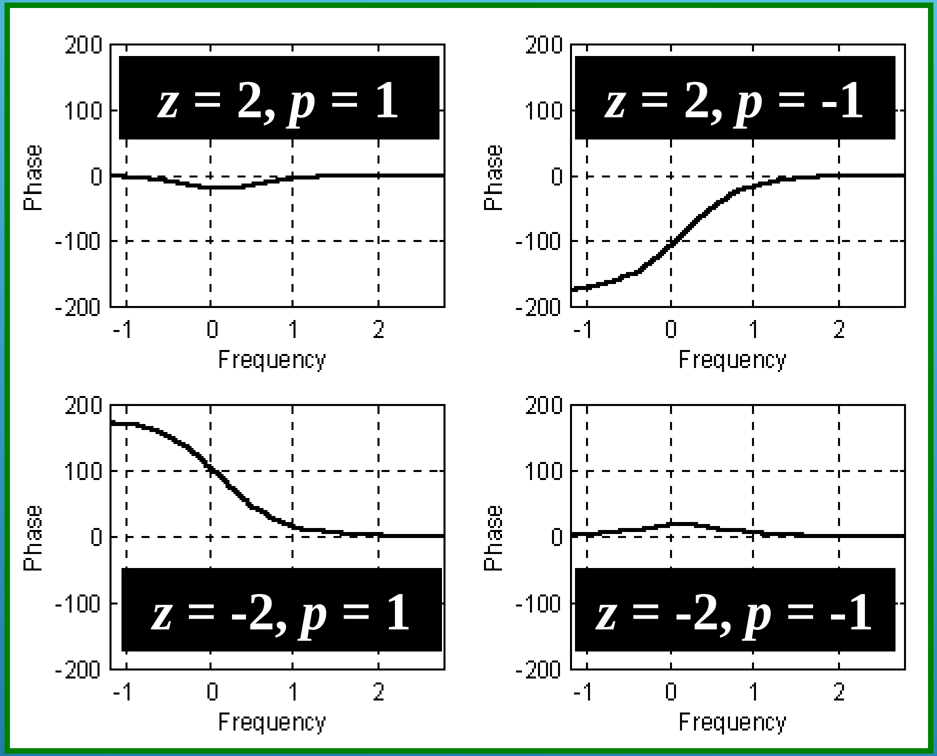
- **Transfer functions having neither poles nor zeros on the right half s-plane are minimum-phase systems.**
- **Transfer functions having poles and/or zeros on the right half s-plane are nonminimum-phase systems.**

Frequency Response Analysis
Minimum-Phase/Nonminimum-Phase Systems

$$G(s) = \frac{s+z}{s+p} \qquad |G(j\omega)| = \sqrt{\frac{\omega^2 + z^2}{\omega^2 + p^2}}$$

z	p	$\lvert G(j\omega) \rvert$	$\angle G(j\omega)$
$+2$	$+1$	$(\omega^2+4)/(\omega^2+1)$	$\arctan(\omega/2) - \arctan(\omega)$
$+2$	-1	$(\omega^2+4)/(\omega^2+1)$	$\arctan(\omega/2) + \arctan(\omega)$
-2	$+1$	$(\omega^2+4)/(\omega^2+1)$	$-(\arctan(\omega/2) + \arctan(\omega))$
-2	-1	$(\omega^2+4)/(\omega^2+1)$	$-(\arctan(\omega/2) - \arctan(\omega))$

Frequency Response Analysis
Minimum-Phase/Nonminimum-Phase Systems



Frequency Response Analysis
Transport Lag (Delay)

$$G(s) = e^{-sT}$$

$$G(j\omega) = e^{-j\omega T} = \cos(\omega T) - j\sin(\omega T)$$

$$|G(j\omega)| = 1$$

$$\angle G(j\omega) = -\omega T \text{ (radians) or } -\frac{180}{\pi}\omega T \text{ (degrees)}$$

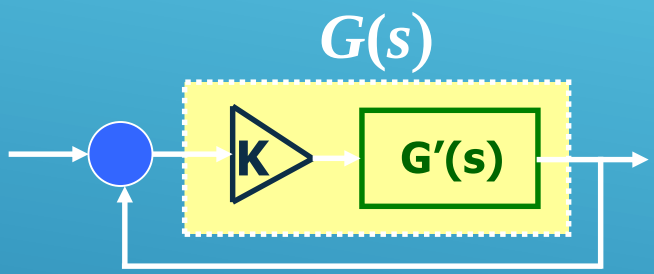
Frequency Response Analysis
Transport Lag - An Example

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

$$\begin{aligned} 20\log\left|G(j\omega)\right| &= 20\log\left|e^{-j\omega L}\right| - 20\log\left|1 + j\omega T\right| \\ &= -20\log\left|1 + j\omega T\right| \end{aligned}$$

$$\angle G(j\omega) = -\omega L - \arctan(\omega T)$$

Frequency Response Analysis
Gain Margin and Phase Margin

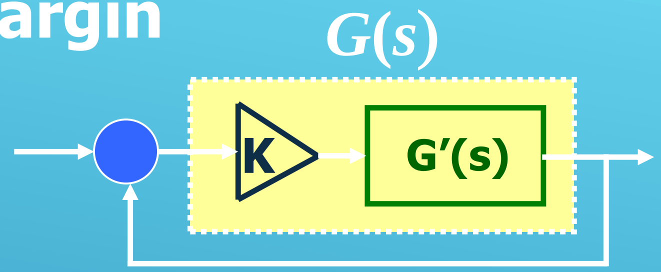


$$T(s) = \frac{G(s)}{1+G(s)} = \frac{KG'(s)}{1+KG'(s)}$$

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} = \frac{KG'(j\omega)}{1+KG'(j\omega)}$$

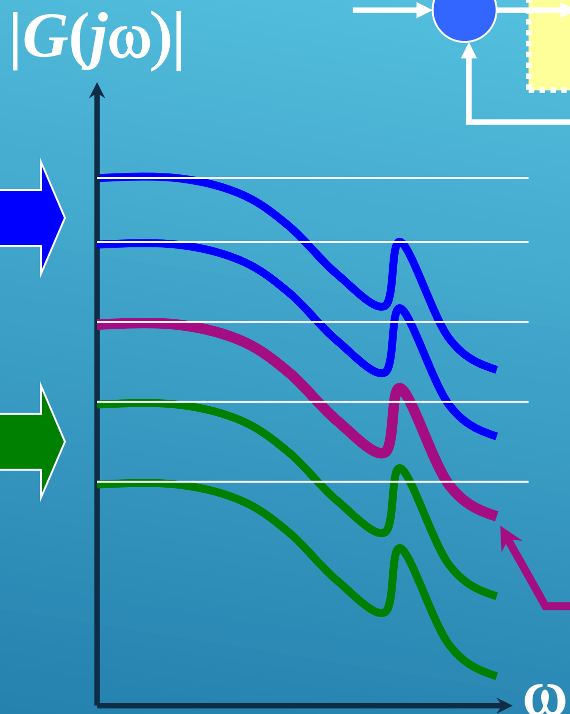
- When **$1+KG'(jw)=0$** holds true, the closed loop system is at the verge of instability.
- At a frequency, say w_1, $G'(jw_1)$ is a negative real number, i.e. $\angle G'(jw_1)=180^\circ$. Then w_1 is called phase crossover frequency. The gain making **$1+K_gG'(jw_1)=0$** is the critical gain, which is the **gain margin** calculated as **$K_g=1/|G'(jw_1)|$**

Frequency Response Analysis
Gain Margin and Phase Margin



Increasing the gain K lifts up the magnitude curve

Decreasing the gain K lowers down the magnitude curve



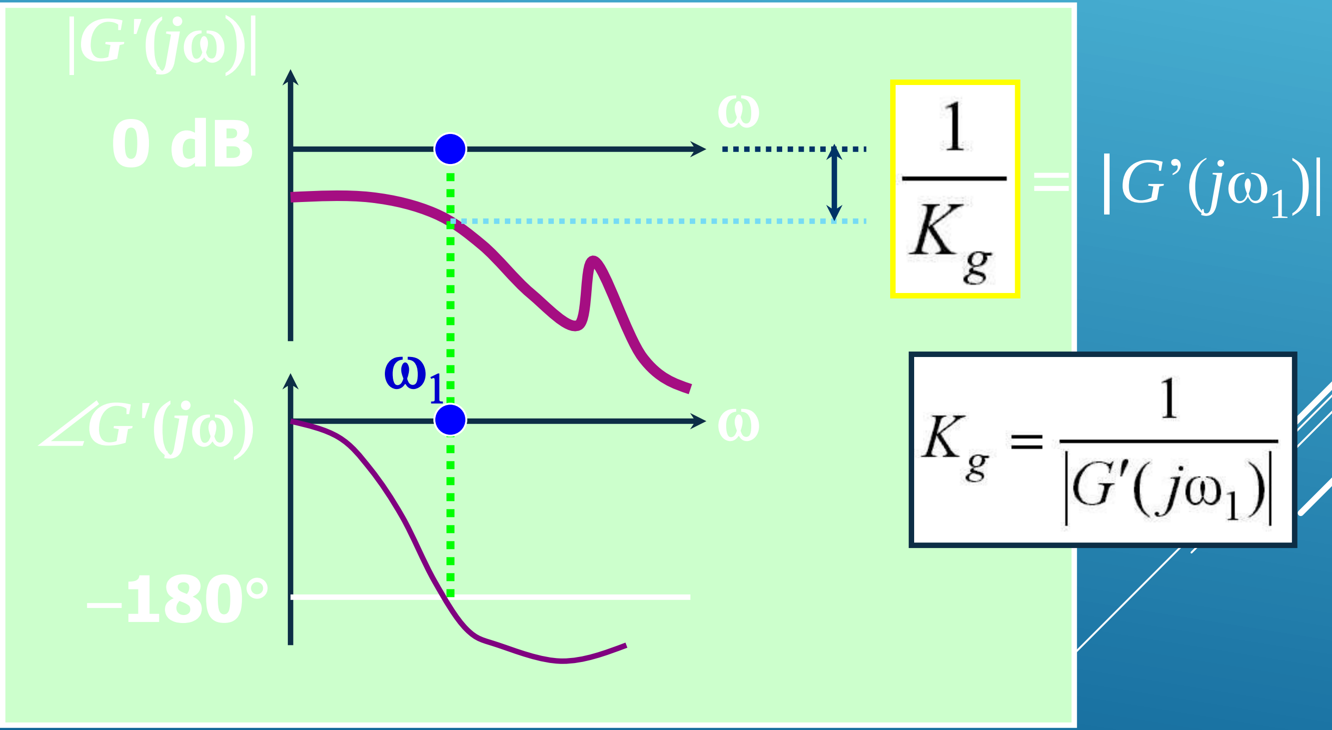
$$T(s) = \frac{G(s)}{1+G(s)}$$

For K=1 Mag. of G'

Clearly, if you change the gain K, the phase curve of G(s) will not be affected.

Frequency Response Analysis
Gain Margin and Phase Margin

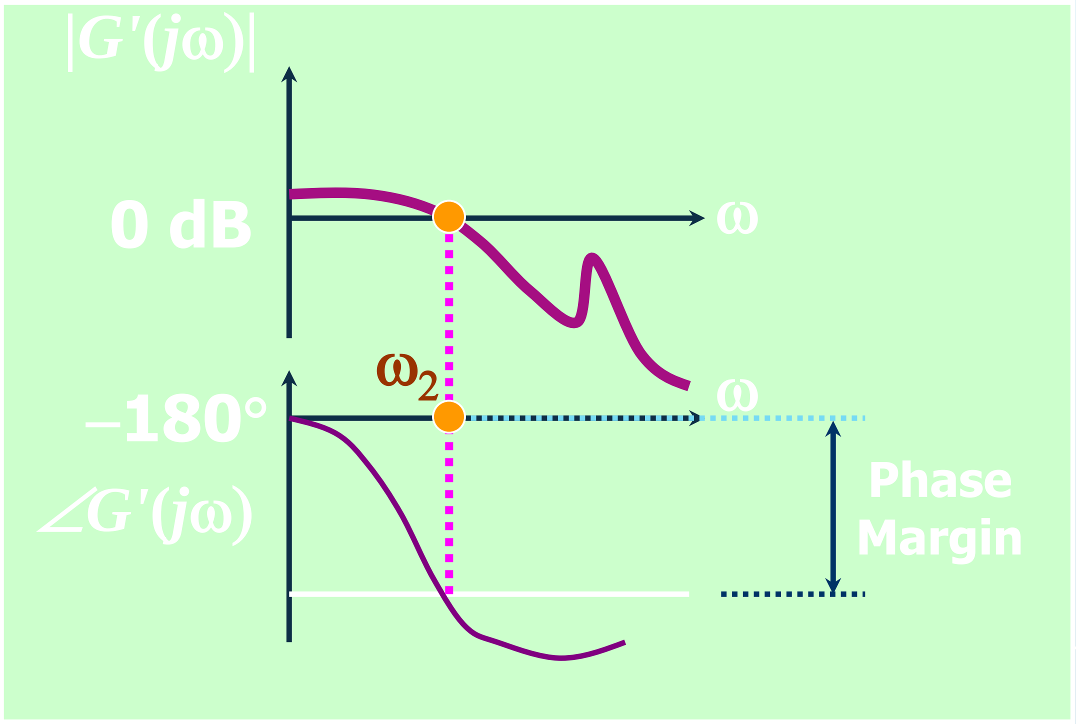
- **Find the smallest frequency (the phase crossover frequency, ω_1) at which the phase angle of the open loop TF is –180°. Note that the phase curve of G(s) is equal to that of G'(s) since K≥0.**



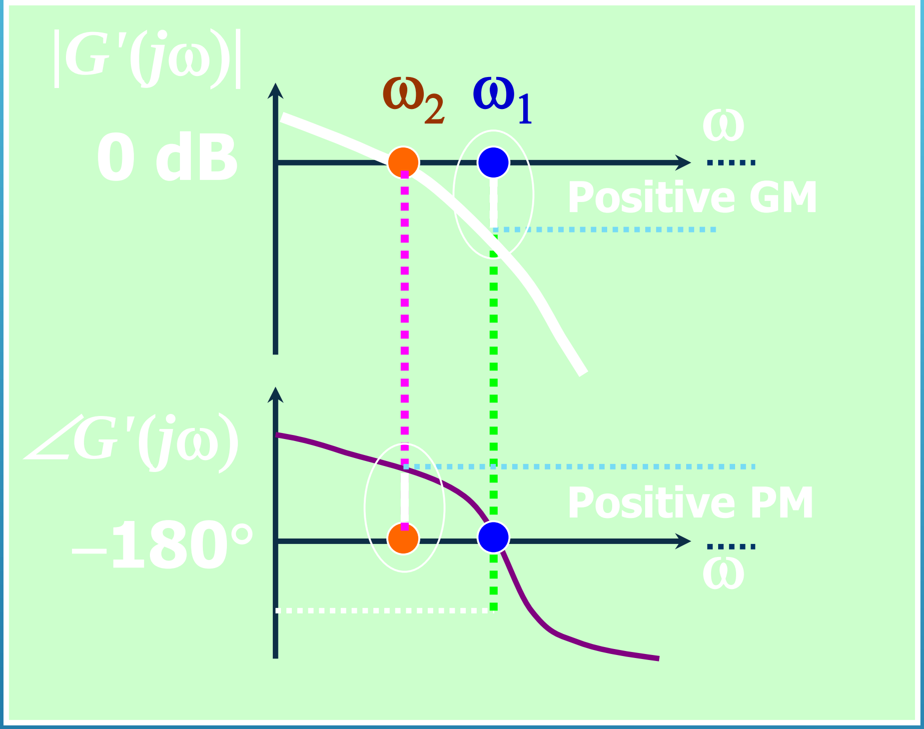
$$K_g = \frac{1}{|G'(j\omega_1)|}$$

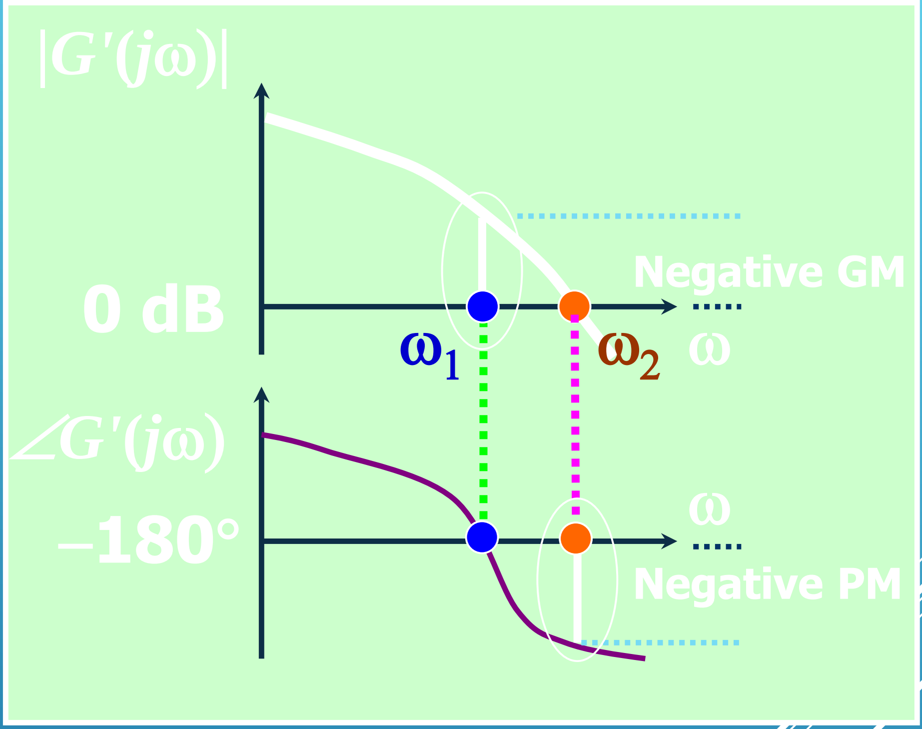
Frequency Response Analysis
Gain Margin and Phase Margin

- **Find the smallest frequency (the gain crossover frequency, ω_2) at which the magnitude of the open loop TF is 0 dB.**



Frequency Response Analysis
Gain Margin and Phase Margin





System is stable
You can multiply the current loop gain at most by K_g

System is unstable!
You have to divide the current loop gain at least by K_g

When you take the logarithm, your action will move [redacted]pwards or downwards.

Frequency Response Analysis
Gain Margin and Phase Margin

- **Can I find the same upper limit of gain by using Routh criterion?**

  YES…

- **So, why don't we use it?**

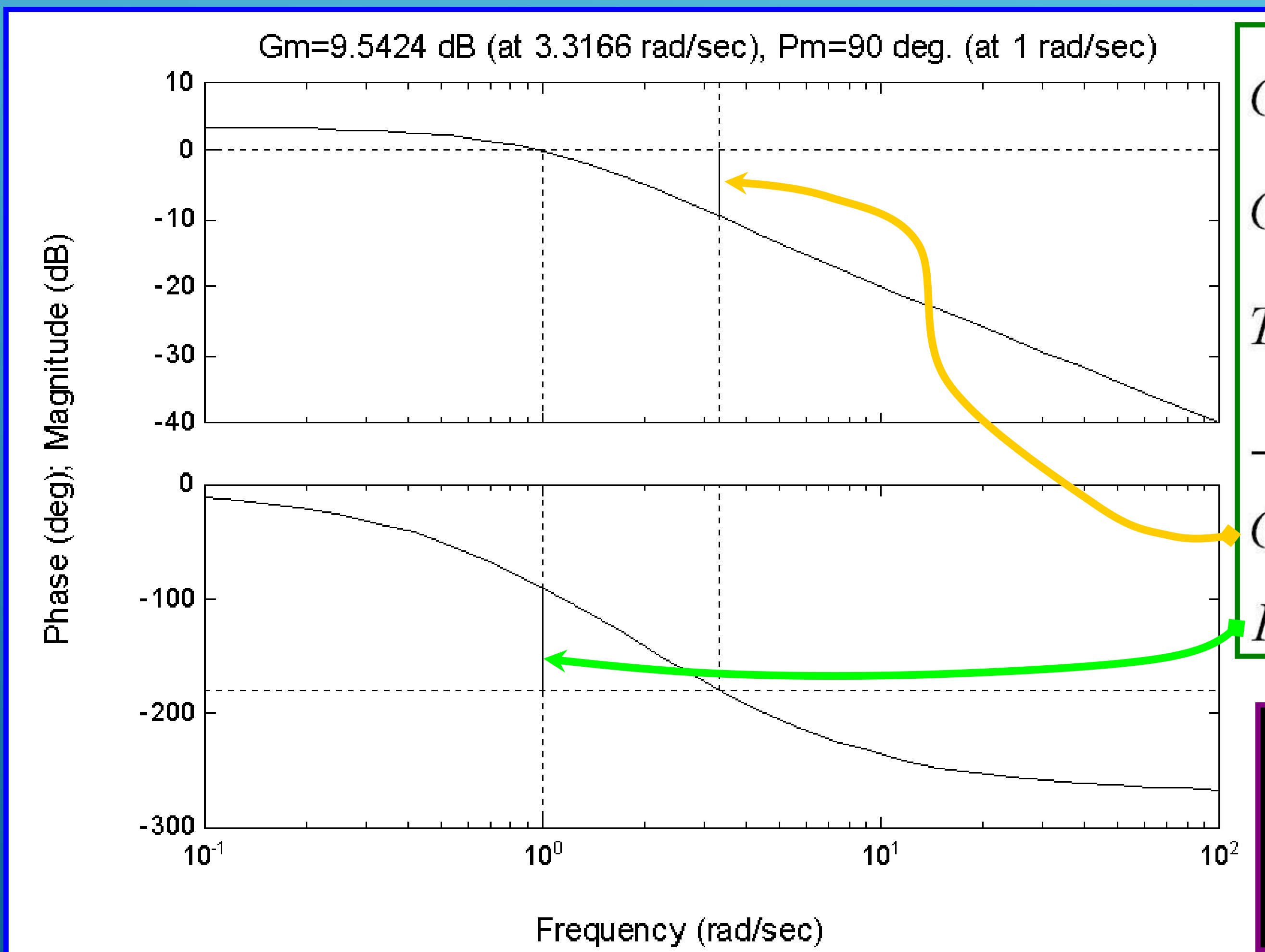
  Routh criterion does not tell anything about relative stability. The quantity 1+KG′(jw) for a fairly valid K may be very close to zero in magnitude! A tiny variation in G′(jw) might let you troubled then…

- **Is this the only way to study relative stability?**

  No. We will see Nyquist plots and draw the parallels between gain margin & phase margin and Nyquist curve.

[redacted]

Frequency Response Analysis
Gain Margin and Phase Margin
An Illustrative Example



$$G'(s) = \frac{3-s}{s^2+3s+2}$$

$$G(s) = KG'(s)$$

$$T(s) = \frac{G(s)}{1+G(s)}$$

$$-2/3 < K < 3 \ \text{(Routh Cr.)}$$

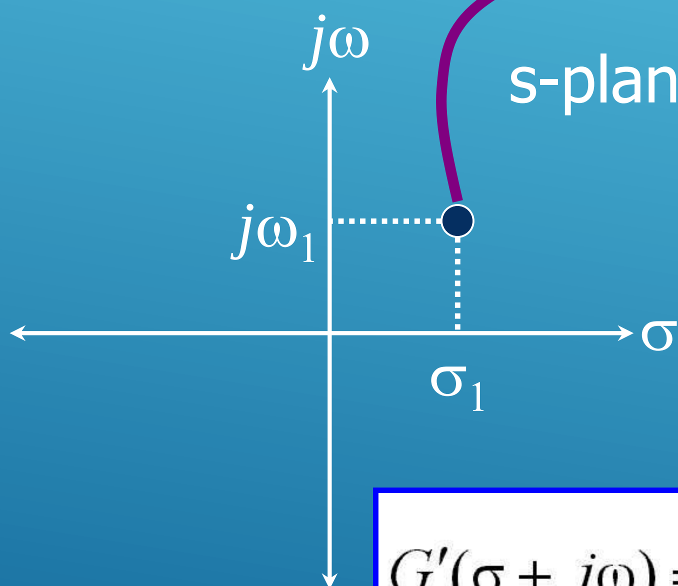
$$GM = 20\log 3 \cong 9.5424$$

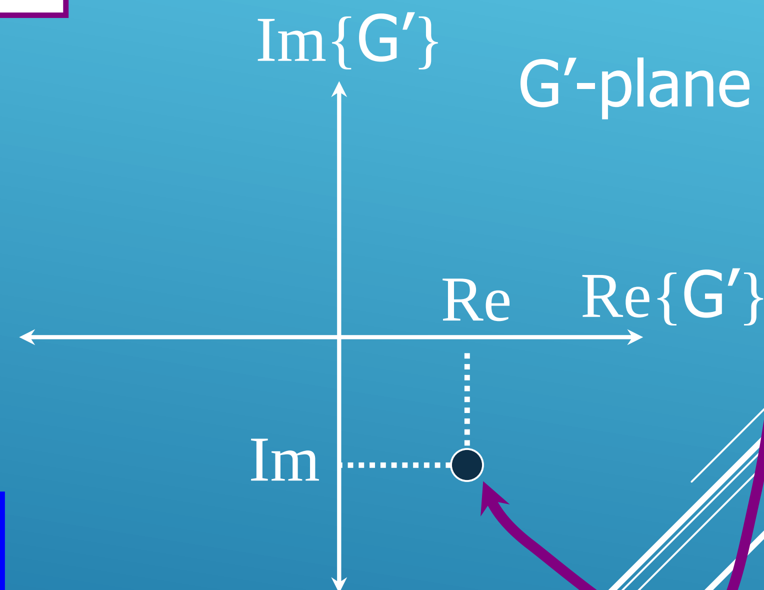
$$PM = 90^\circ \text{ at } 1 \text{ rad/sec}$$

```
num = [-1 3];
den = [1 3 2];
margin(num,den)
```

Frequency Response Analysis
Polar Plots - A Simple Example

$$G'(s) = \frac{2}{s+1}$$



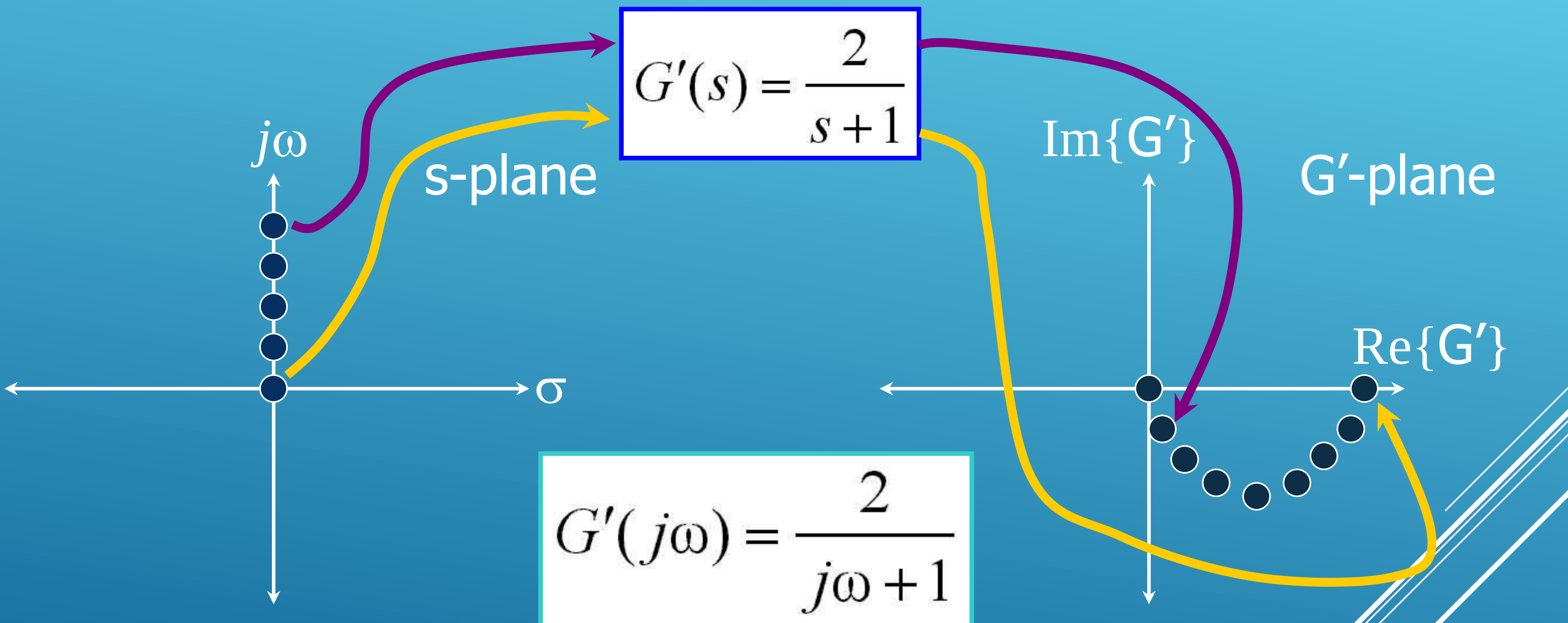


$$G'(\sigma + j\omega) = \frac{2}{(\sigma + j\omega) + 1}$$

$$\mathrm{Re}\{G'\} = \frac{2(\sigma+1)}{(\sigma+1)^2 + \omega^2} \qquad \mathrm{Im}\{G'\} = j\frac{-2\omega}{(\sigma+1)^2 + \omega^2}$$

[redacted]

Frequency Response Analysis
Polar Plots - A Simple Example

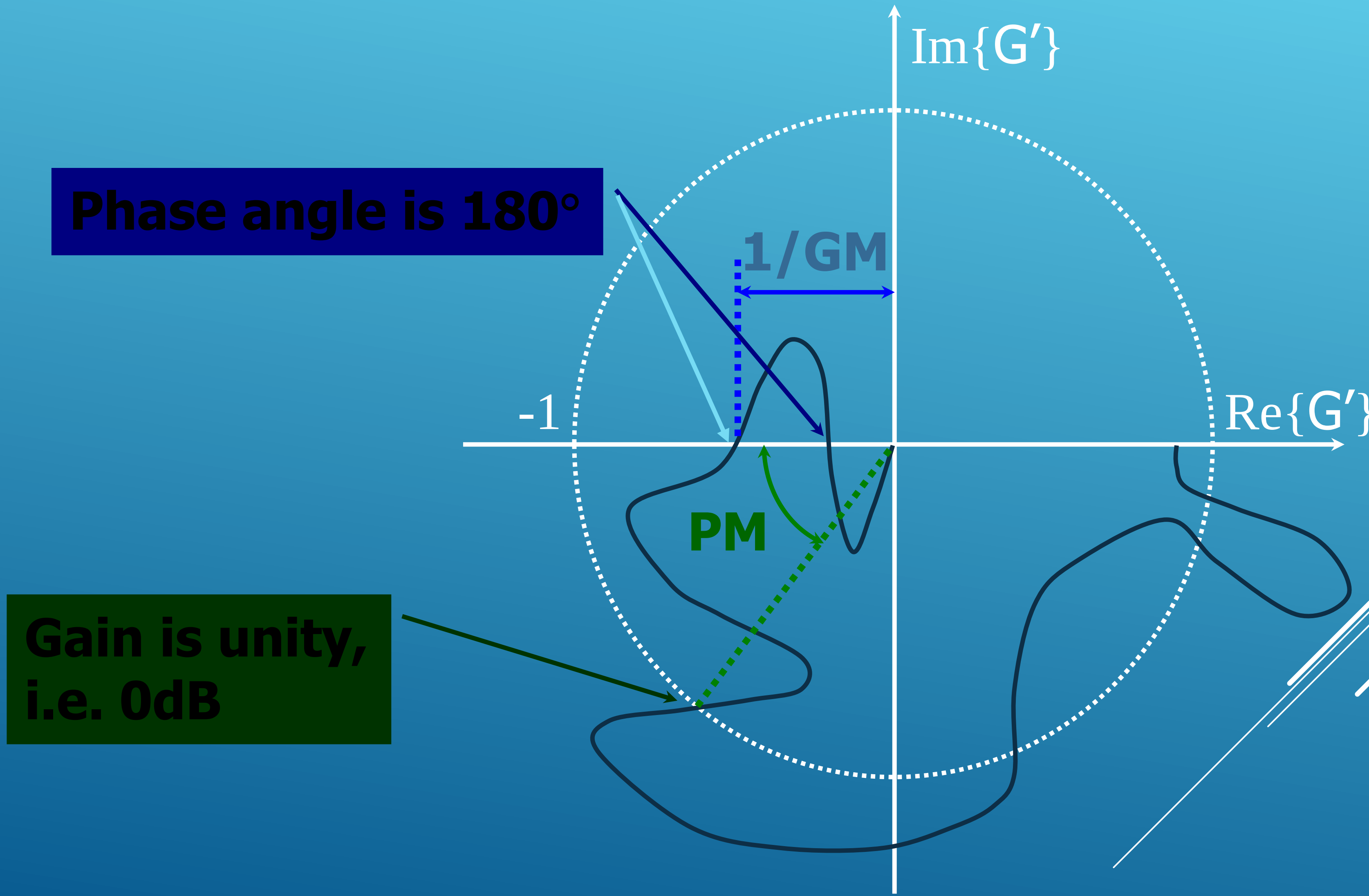


$$G'(s) = \frac{2}{s+1}$$

$$G'(j\omega) = \frac{2}{j\omega+1}$$

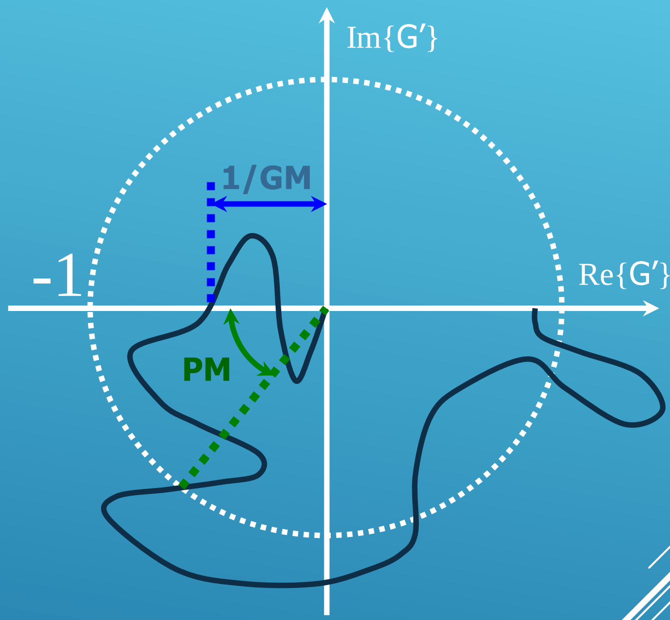
What would you get if you choose all points on the nonnegative part of the imaginary axis?

Frequency Response Analysis
Polar Plots and Margins - A Formal View

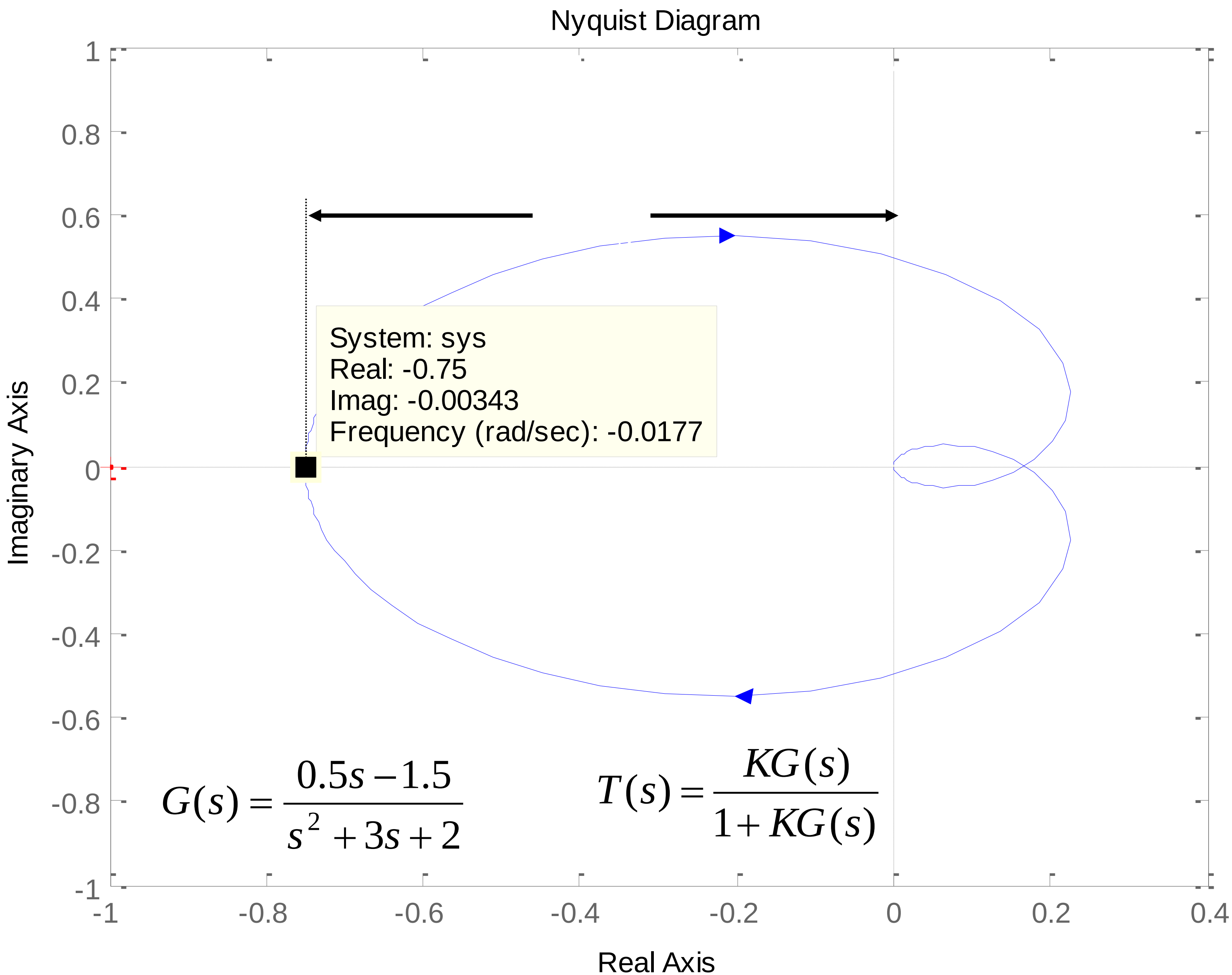


Frequency Response Analysis
Polar Plots and Margins - A Formal View

There may be more than one phase or gain crossover frequencies. We will restrict ourselves to the cases illustrated here.

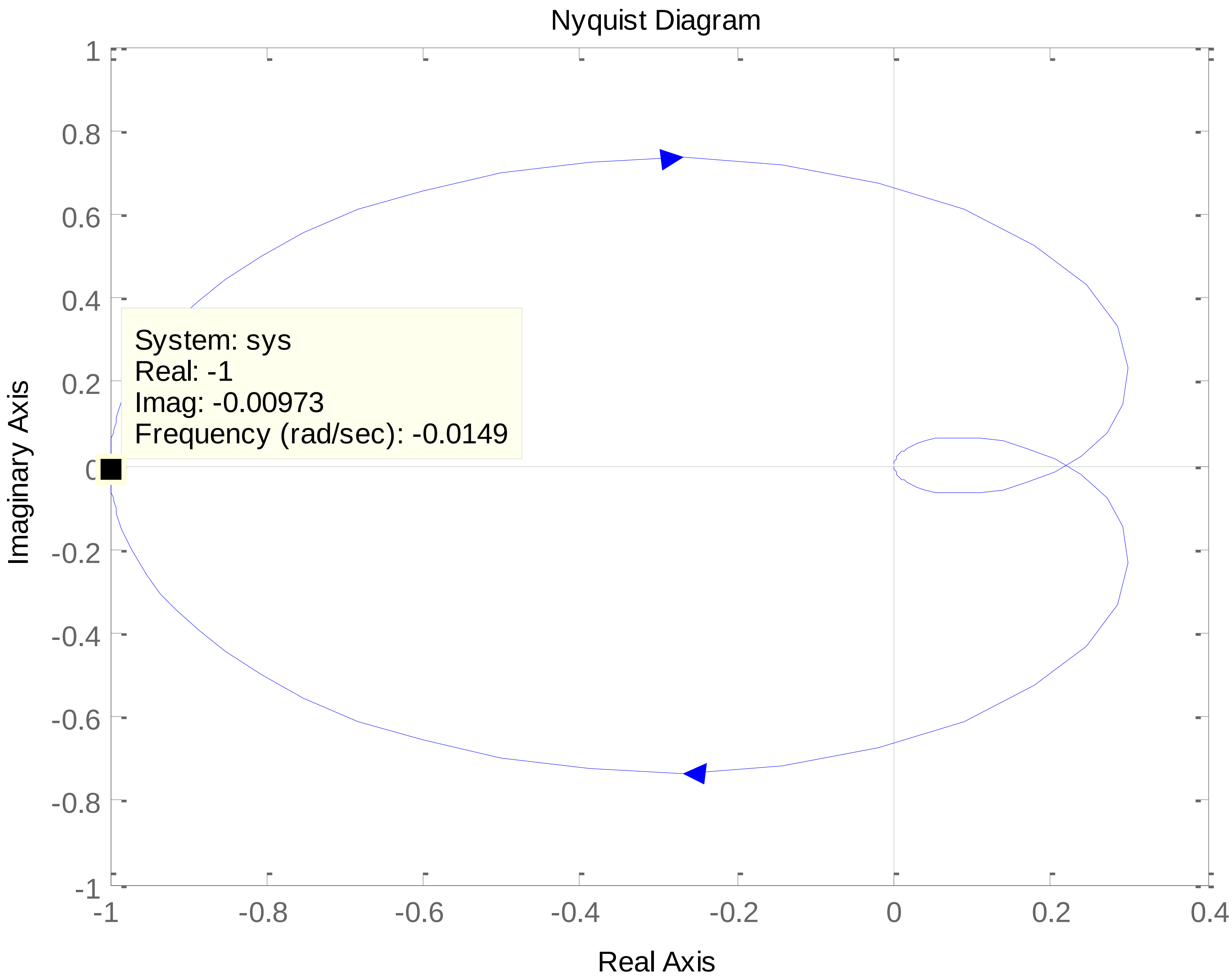


A good discussion on these issues is presented in:
Hitay Özbay, *Introduction to Feedback Control Theory*,
ISBN: 0-8493-1867-X (pp.85-100)

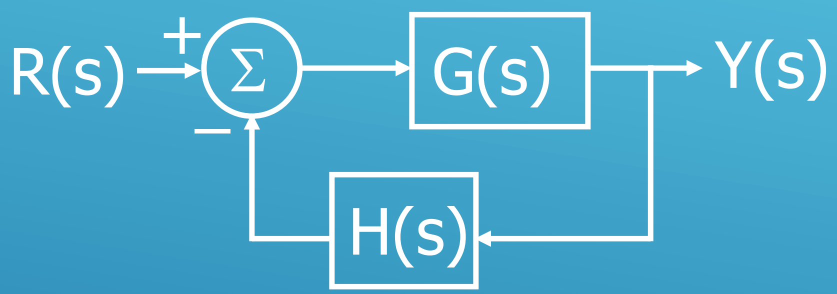


$$G(s) = \frac{0.5s - 1.5}{s^2 + 3s + 2}$$

$$T(s) = \frac{KG(s)}{1 + KG(s)}$$


Nyquist Diagram
System: sys
Real: -1
Imag: -0.00973
Frequency (rad/sec): -0.0149
Imaginary Axis
Real Axis
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

Frequency Response Analysis
Nyquist Stability Criterion



$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

1+G(s)H(s)=0 is the characteristic equation. Nyquist stability criterion lets us know
The number of right half s-plane poles of T(s) by using
The number of right half s-plane poles of G(s)H(s) and
The number of clockwise encirclements of the point -1+j0 made by the polar plot of G(jw)H(jw).
Let's see the details…

Frequency Response Analysis
Nyquist Stability Criterion

Why are we interested in the point -1+j0 ?

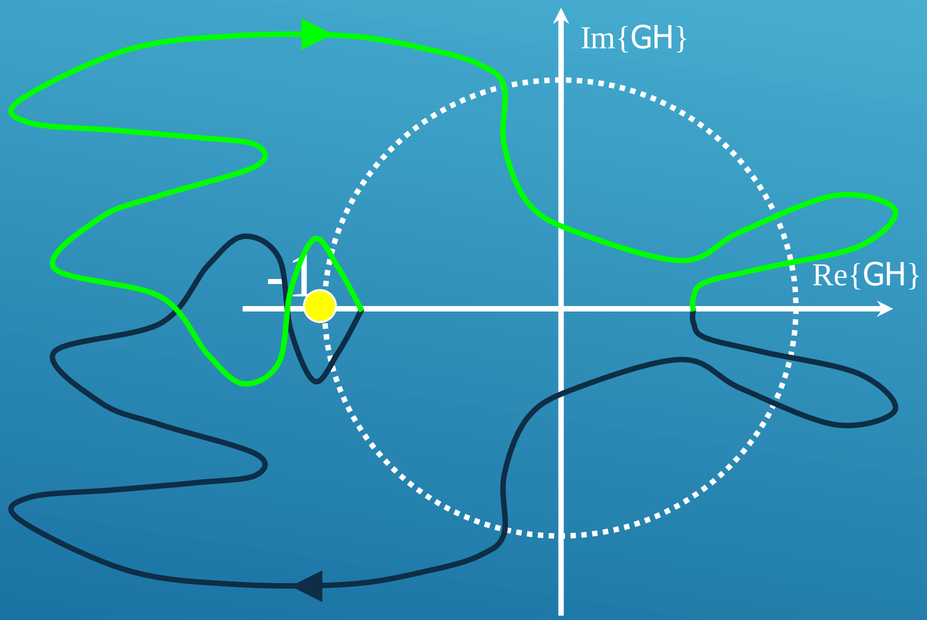
Because the denominator of

$$T(s) = \frac{G(s)}{1+G(s)H(s)}$$

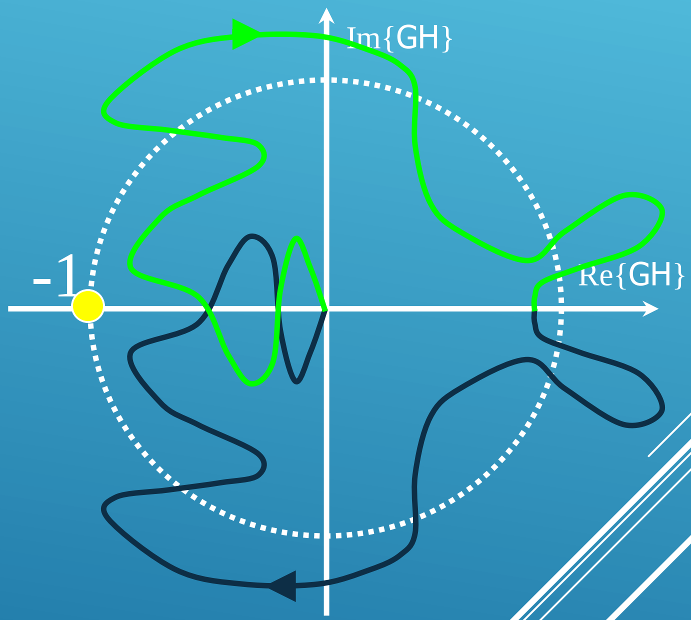
is equal to zero when G(s)H(s)=-1=-1+j0.
Let s=jw, and obtain the polar plot of G(jw)H(jw) while running w from 0 to ∞.
Intuitively, we can say that the closed loop poles should somehow be related to the deployment of the geometric place of G(jw)H(jw) curve according to point -1+j0.

Frequency Response Analysis
Nyquist Stability Criterion

- **What is encirclement?**



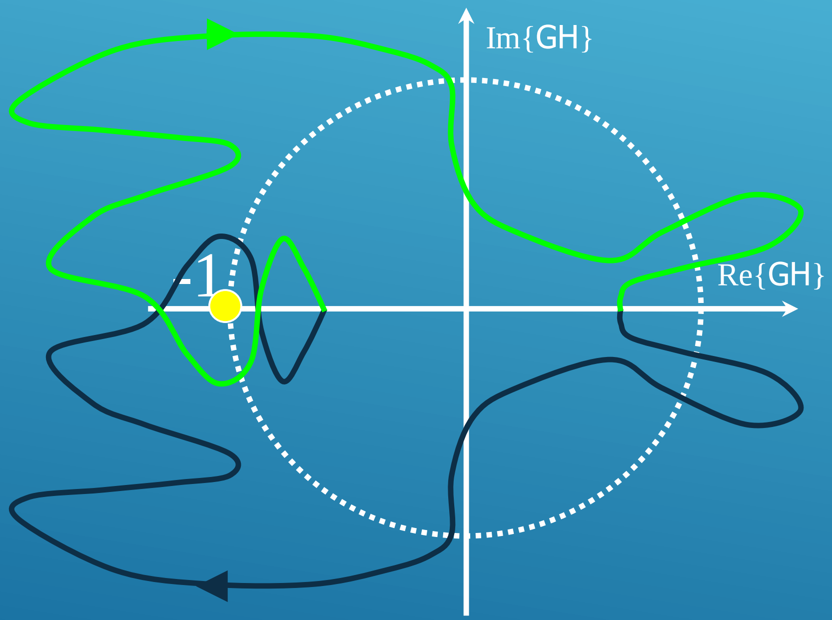
No encirclement!

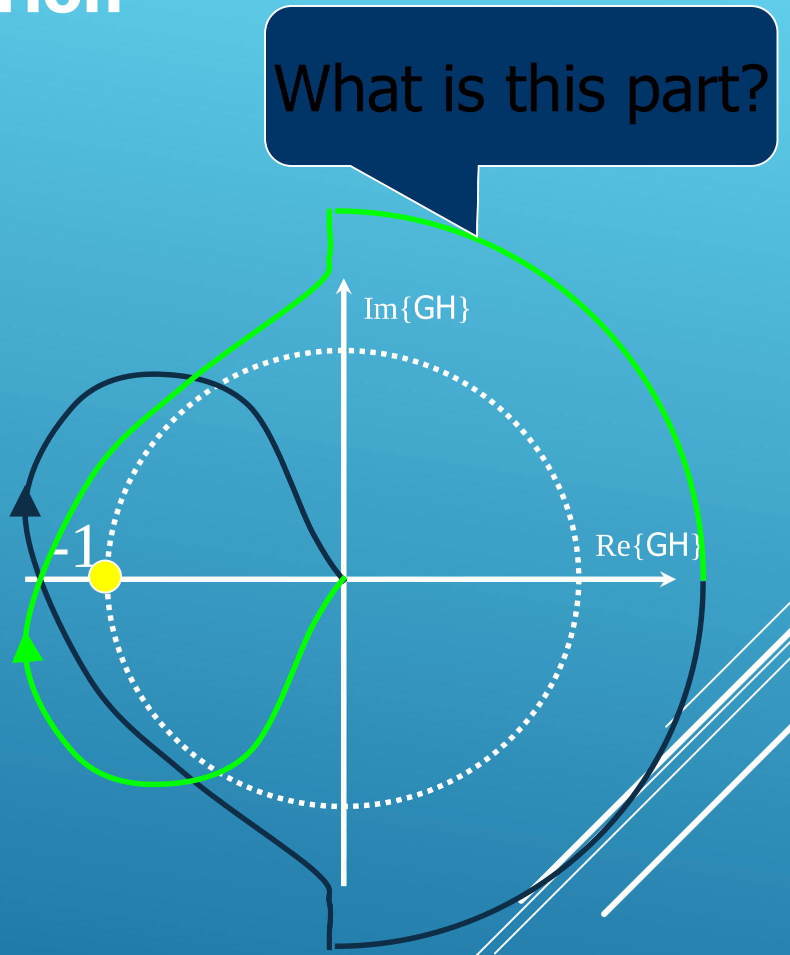


No encirclement

Frequency Response Analysis
Nyquist Stability Criterion

- **What is encirclement?**



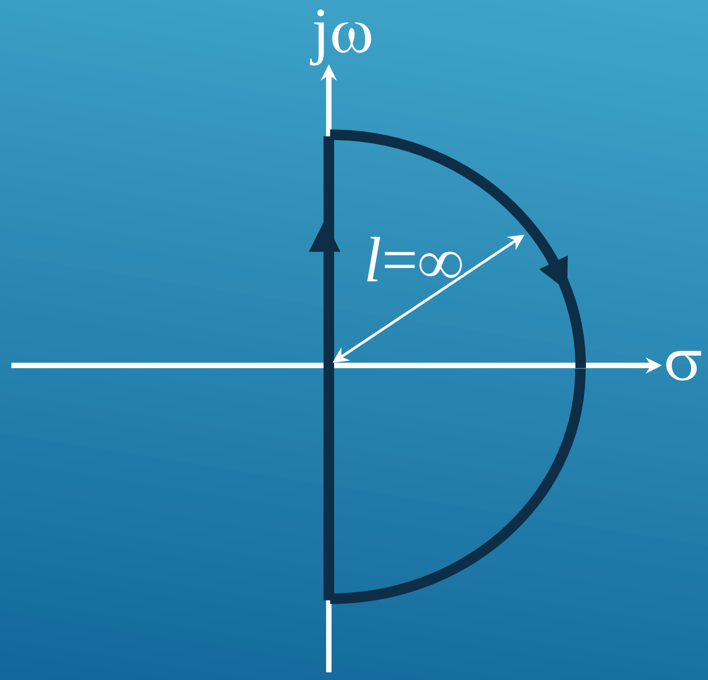


2 clockwise encirclements!

2 clockwise encirclements!

Frequency Response Analysis
Nyquist Stability Criterion

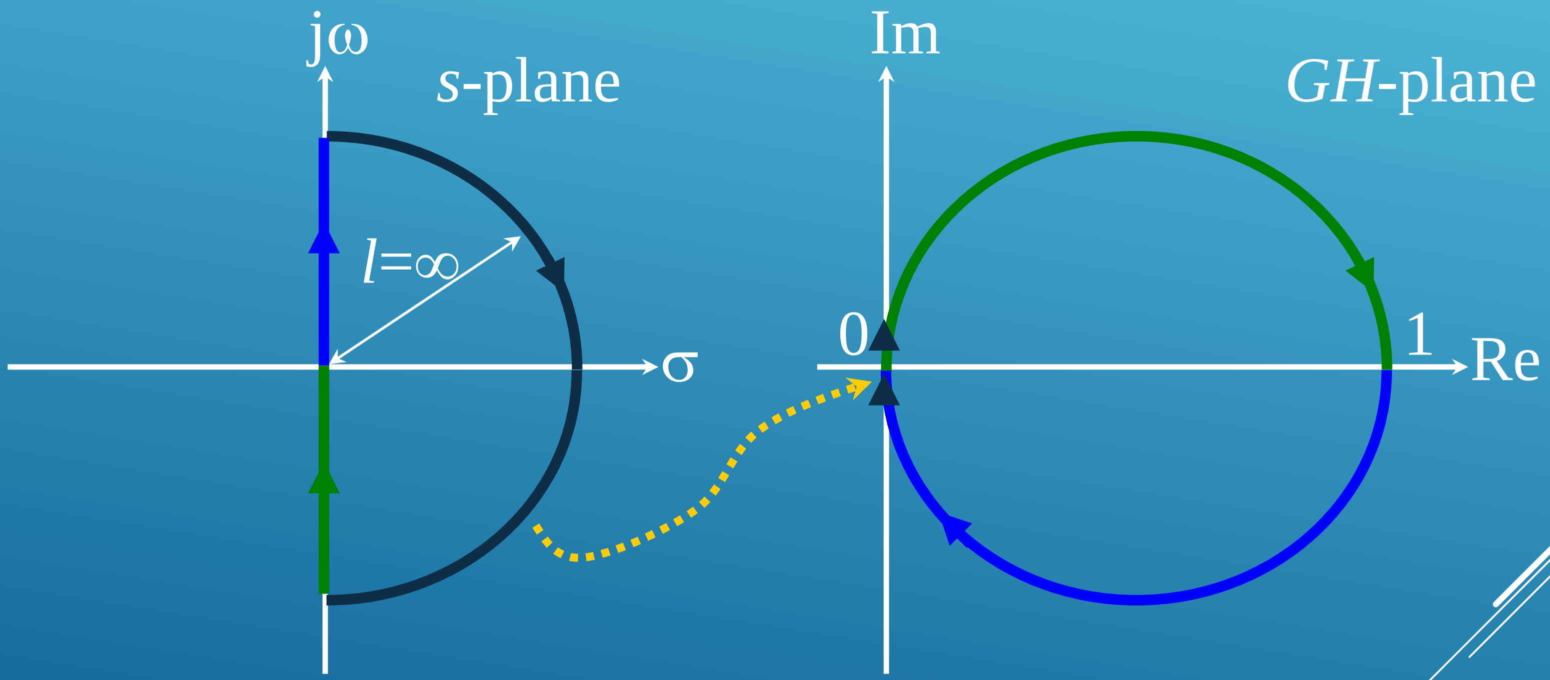
- **Let's see the mapping between a special clockwise contour in s-plane and the curve it corresponds in G(jw)H(jw) plane.**



Since the radius is ∞, the interior of this closed contour contains every unstable zero or pole of the open loop transfer function G(s)H(s), and we can use the theorems of complex mathematics for our goals.

Frequency Response Analysis
Nyquist Stability Criterion

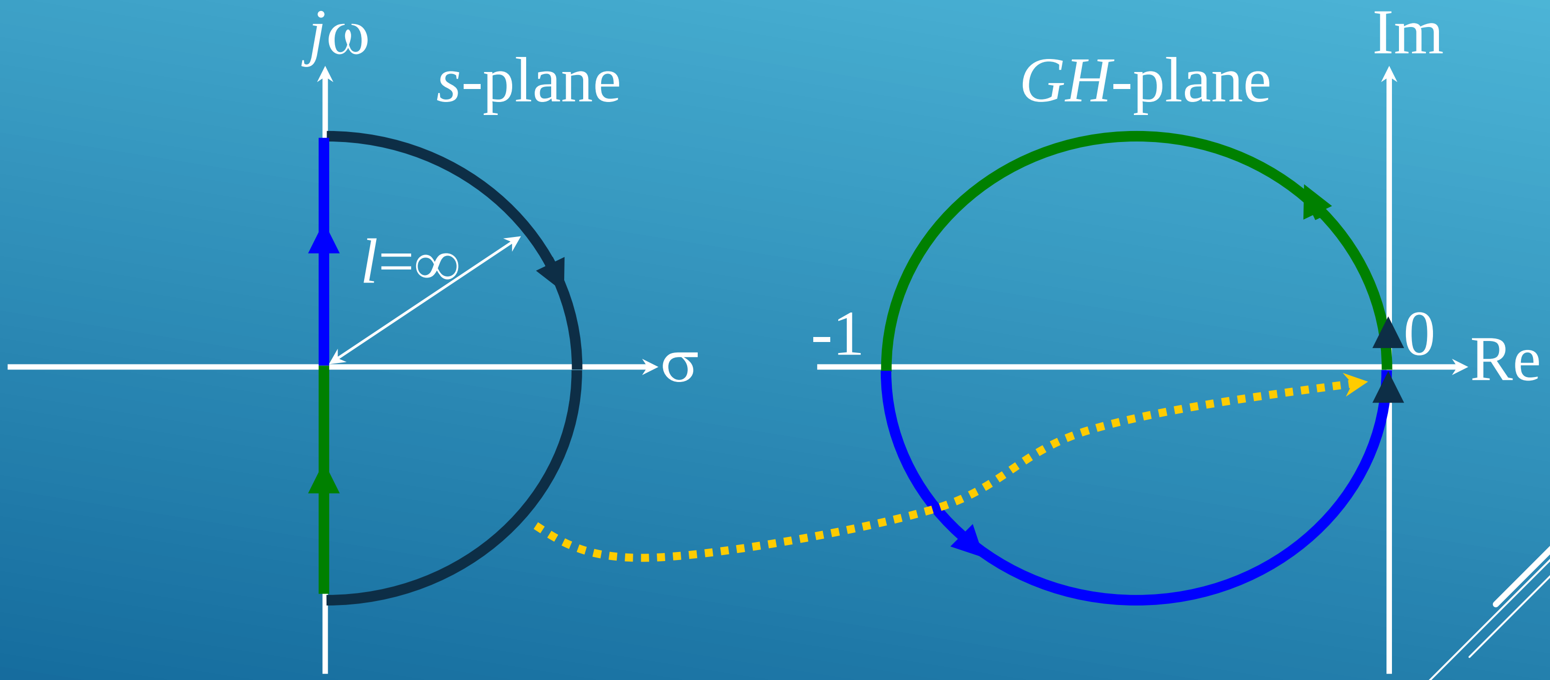
- **G(s)H(s)=1/(s+1), clearly G(jw)H(jw)=1/(jw+1)**



Note that we have not told anything about stability yet! All we are doing now is to see the correspondence.

Frequency Response Analysis
Nyquist Stability Criterion

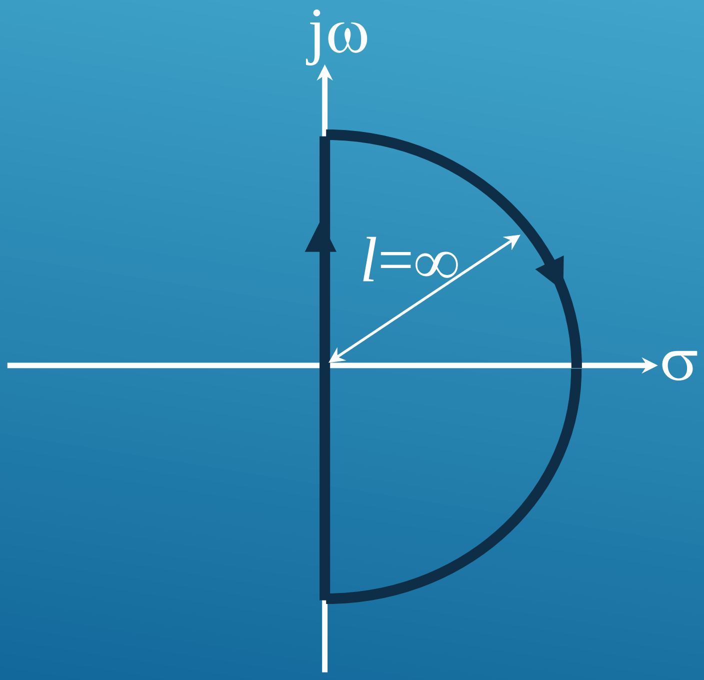
- **G(s)H(s)=1/(s–1), clearly G(jw)H(jw)=1/(jw-1)**



Note that we have not told anything about stability yet! All we are doing now is to see the correspondence.

Frequency Response Analysis
Nyquist Stability Criterion

- G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}

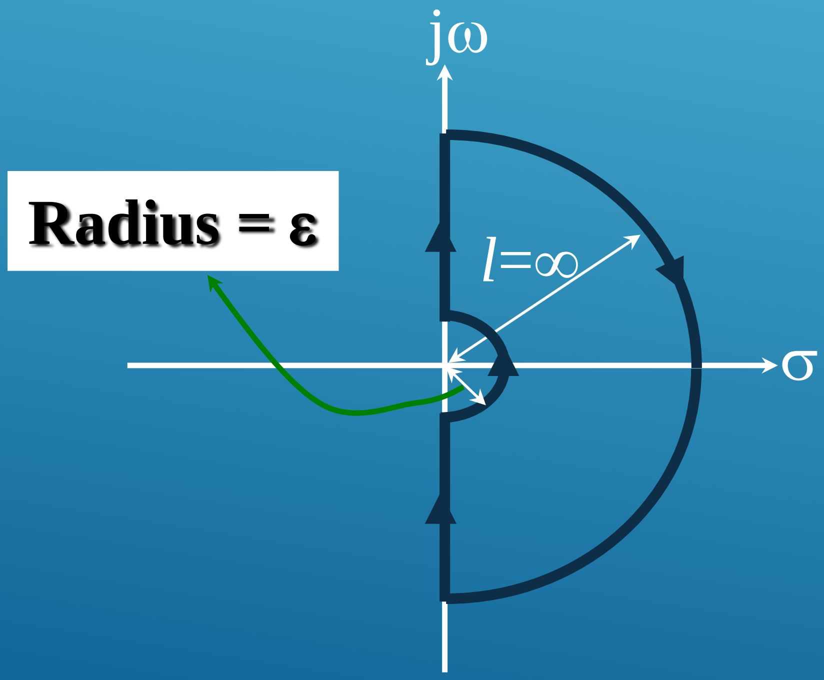


You cannot choose this contour any more! The contour passes through a singularity (There is a pole at s=0).

Detour around it by adding a semicircle of infinitesimal radius ε!

Frequency Response Analysis
Nyquist Stability Criterion

- **G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}**

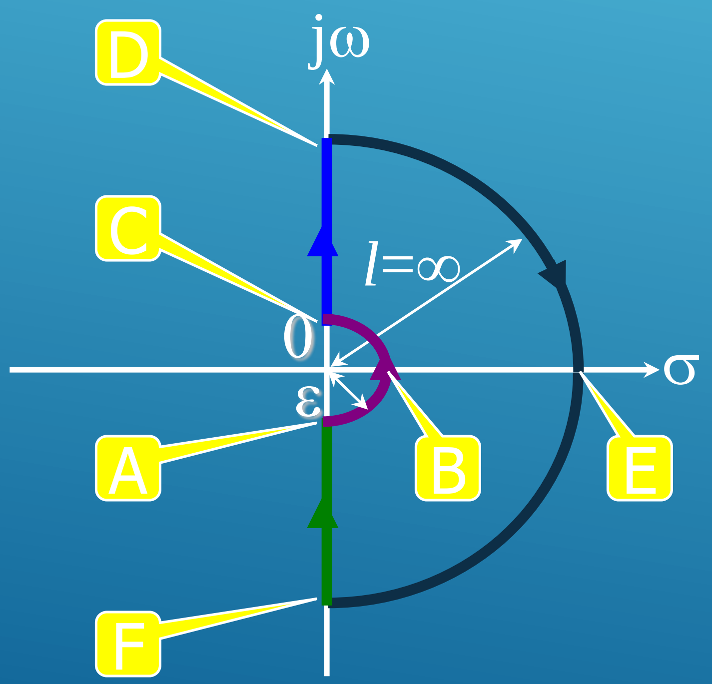


Detour around it by adding a semicircle of infinitesimal radius ε!

Let's analyze what happens now…

Frequency Response Analysis
Nyquist Stability Criterion

- **$G(s)H(s)=1/\{s(s+1)\}$, clearly $G(jw)H(jw)=1/\{jw(jw+1)\}$**



Note that $l \to \infty$ and $\varepsilon \to 0$

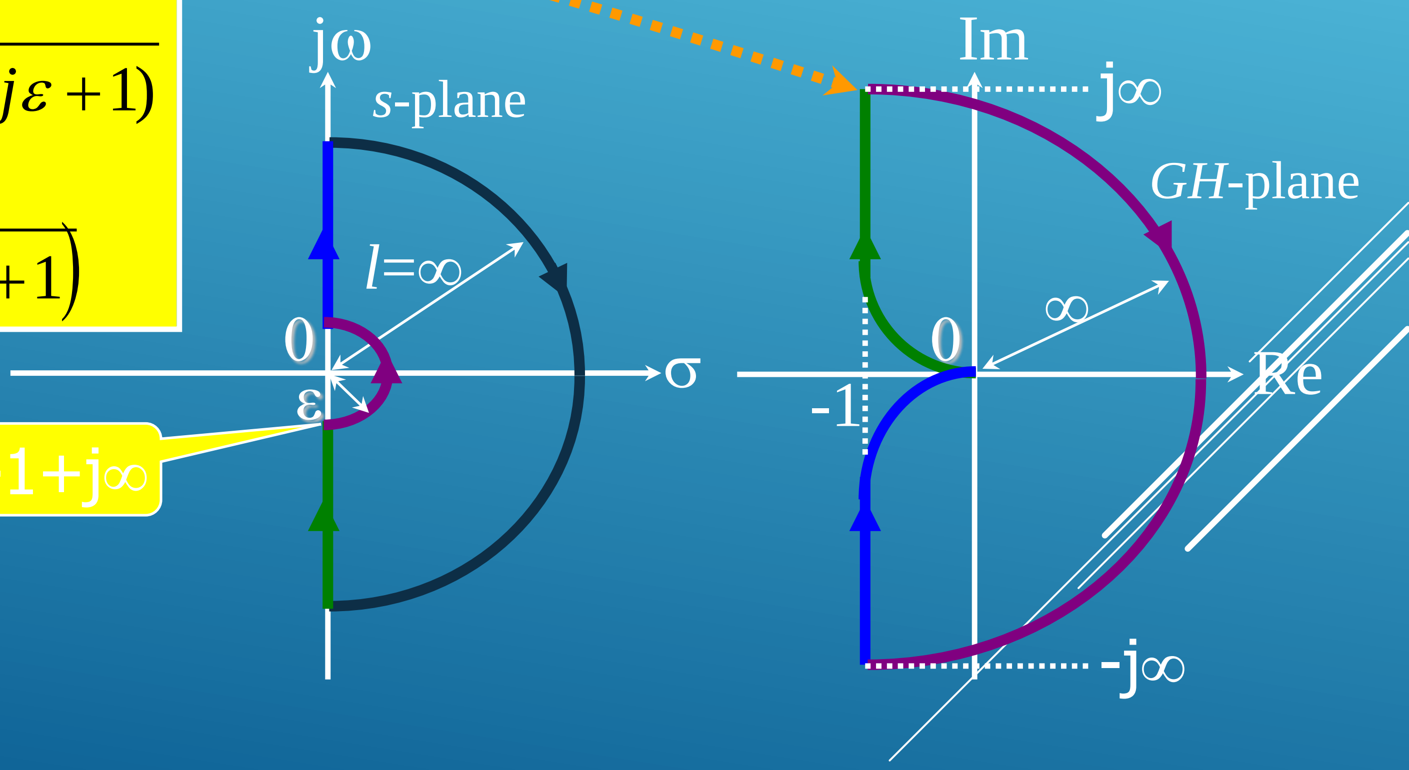
CD	$s=jw$	w:from ε to l
DEF	$s=le^{j\theta}$	θ: from $\pi/2$ to $-\pi/2$
FA	$s=jw$	w:from $-l$ to $-\varepsilon$
ABC	$s=\varepsilon e^{j\theta}$	θ: from $-\pi/2$ to $\pi/2$

Frequency Response Analysis
Nyquist Stability Criterion

- **G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}**

$$\frac{1}{s(s+1)} = \frac{1}{-j\varepsilon(-j\varepsilon+1)}$$
$$= \frac{-1}{\varepsilon^2+1} + j\frac{1}{\varepsilon\left(\varepsilon^2+1\right)}$$

A: G(jw)H(jw)=-1+j∞

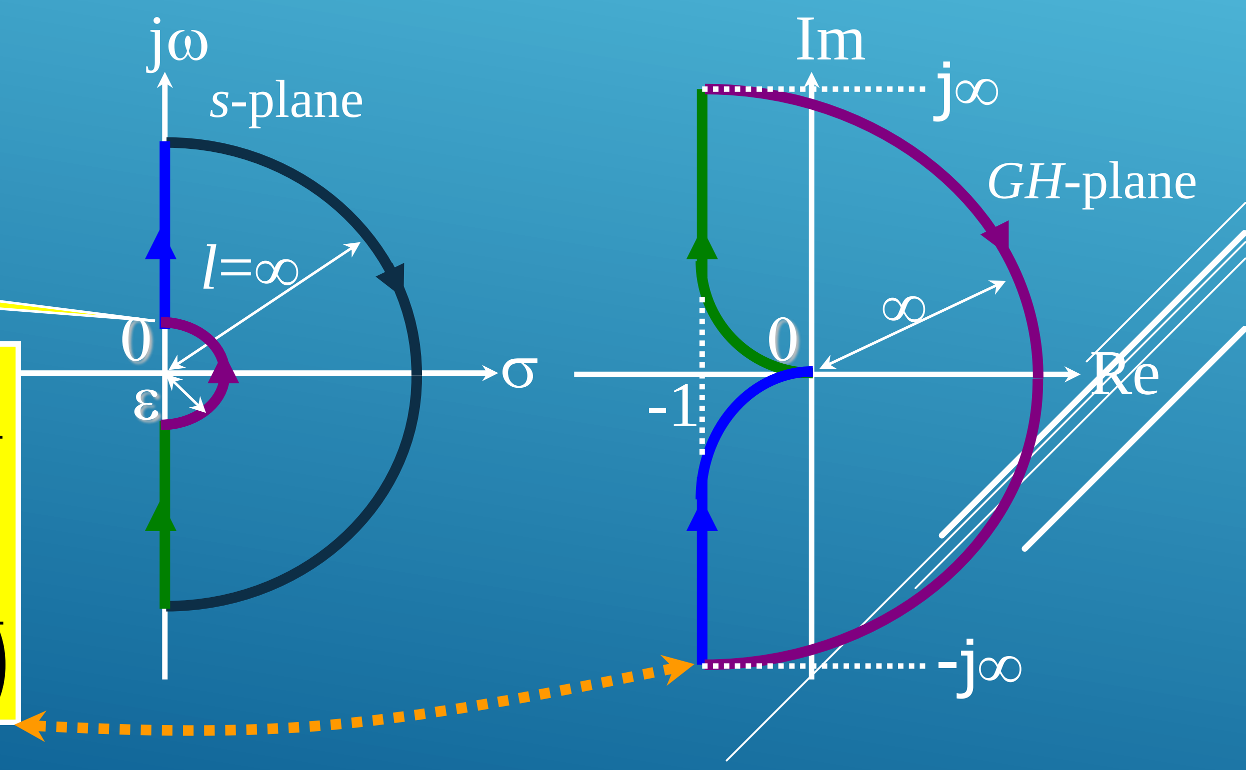


Frequency Response Analysis
Nyquist Stability Criterion

- **G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}**

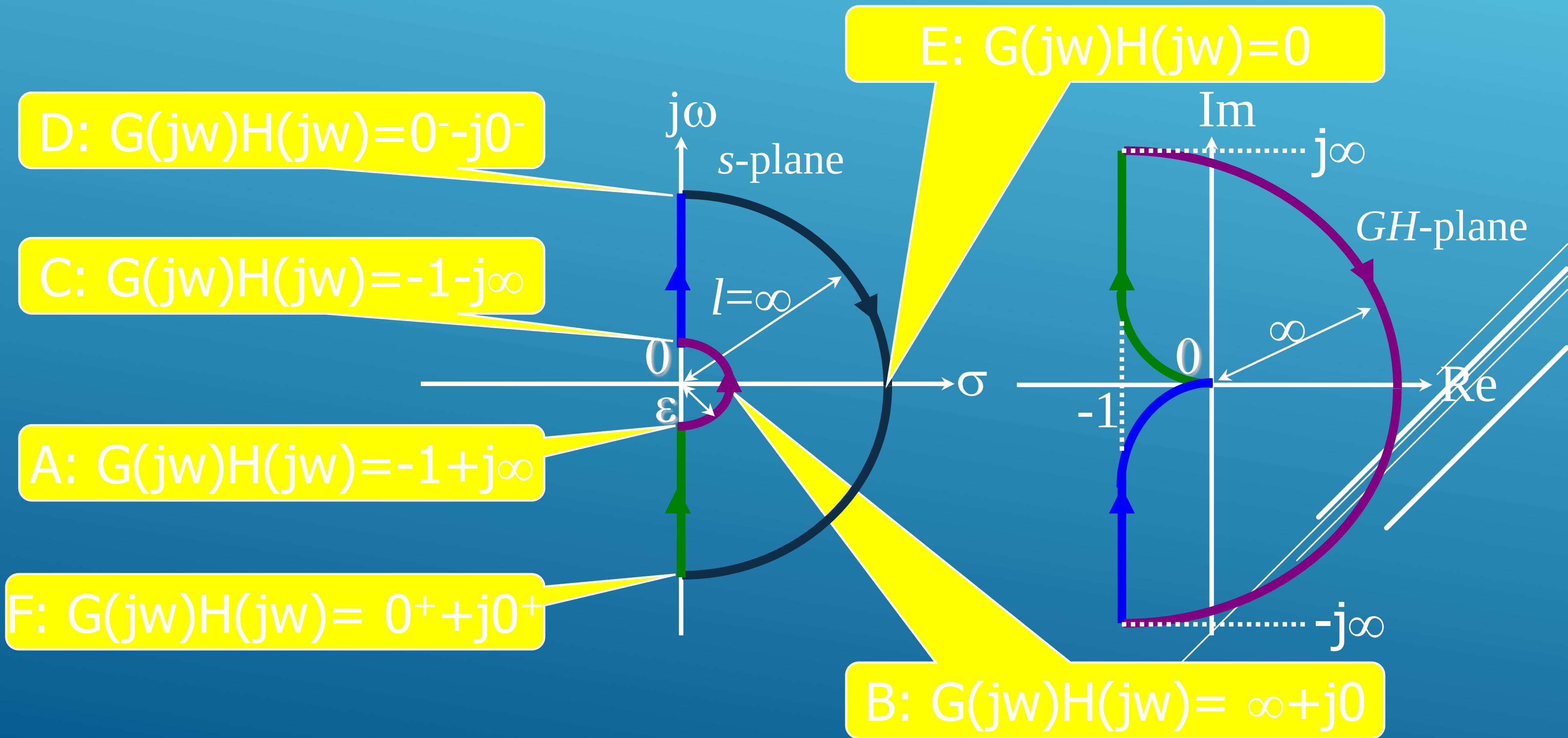
C: G(jw)H(jw)=-1-j∞

$$\frac{1}{s(s+1)}=\frac{1}{j\varepsilon(j\varepsilon+1)}$$
$$=\frac{-1}{\varepsilon^2+1}-j\frac{1}{\varepsilon\left(\varepsilon^2+1\right)}$$



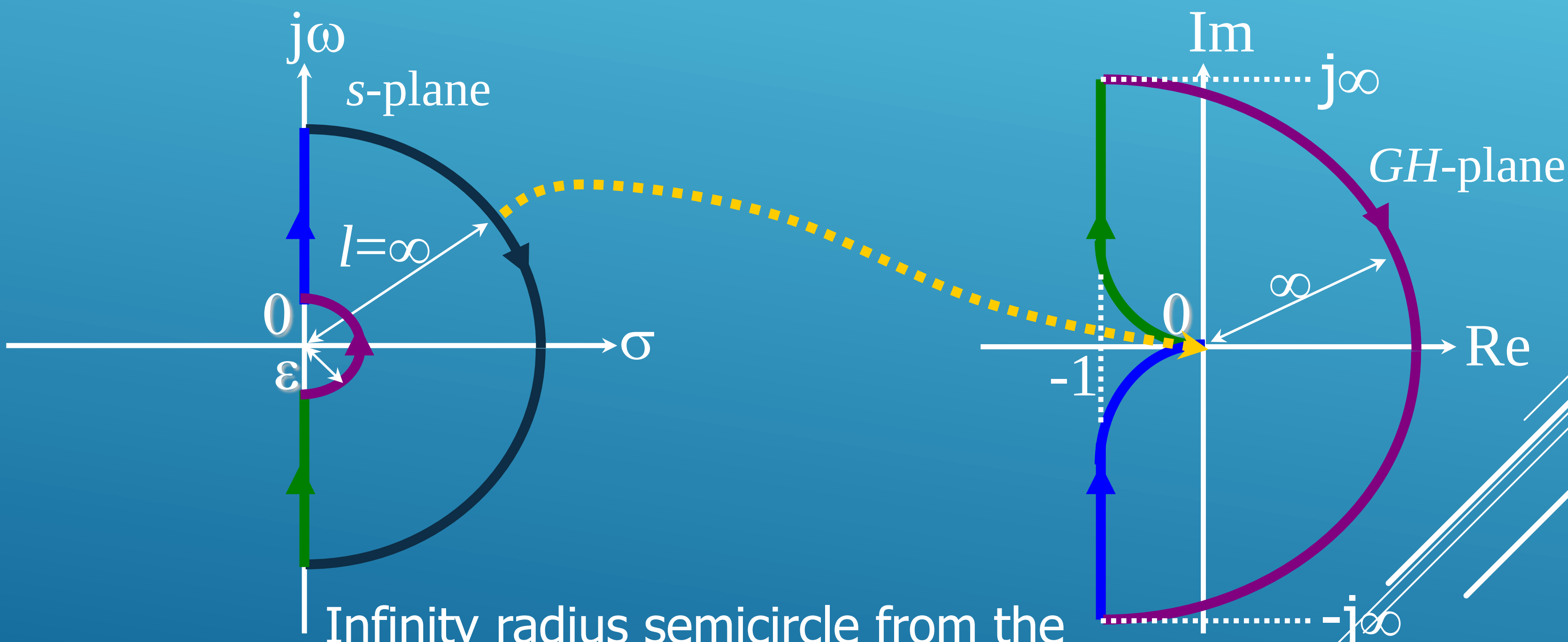
Frequency Response Analysis
Nyquist Stability Criterion

- **$G(s)H(s)=1/\{s(s+1)\}$, clearly $G(jw)H(jw)=1/\{jw(jw+1)\}$**



Frequency Response Analysis
Nyquist Stability Criterion

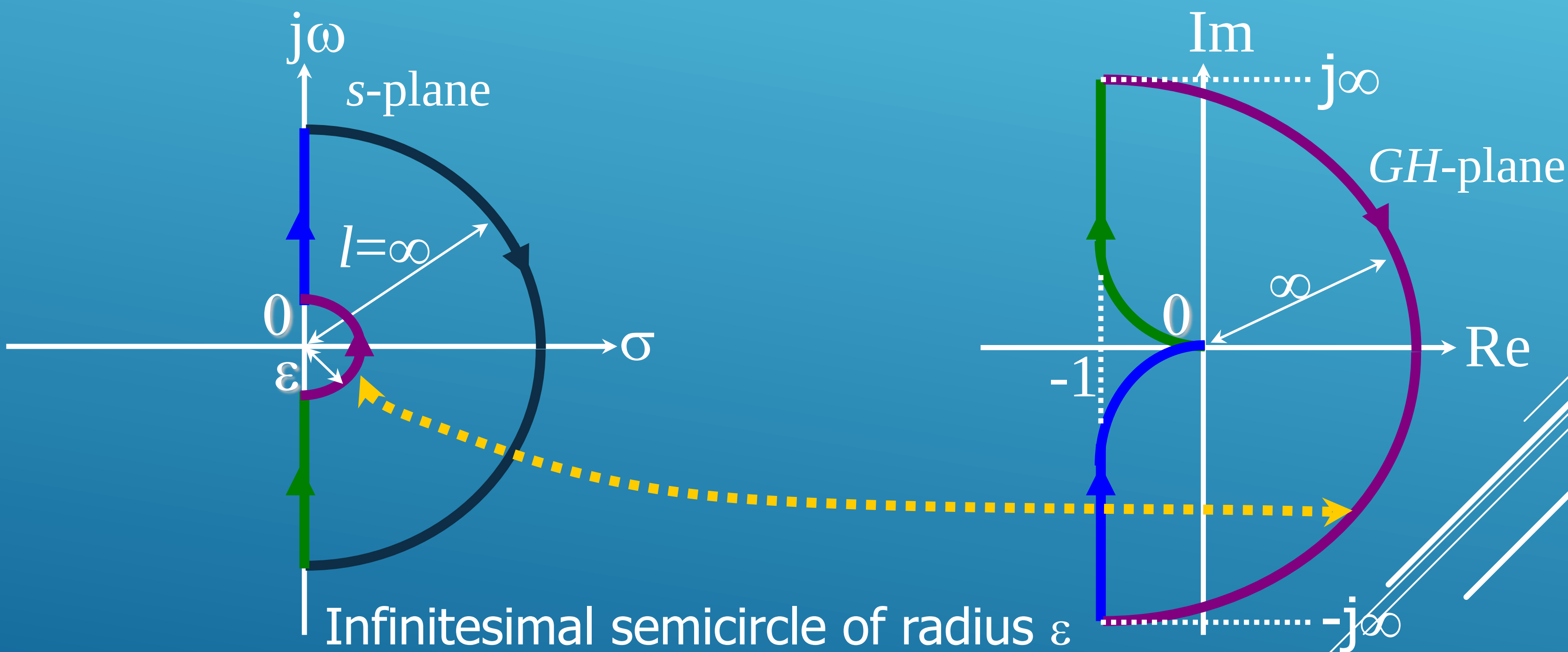
- **G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}**



Infinity radius semicircle from the s-plane is mapped to the origin of GH-plane

Frequency Response Analysis
Nyquist Stability Criterion

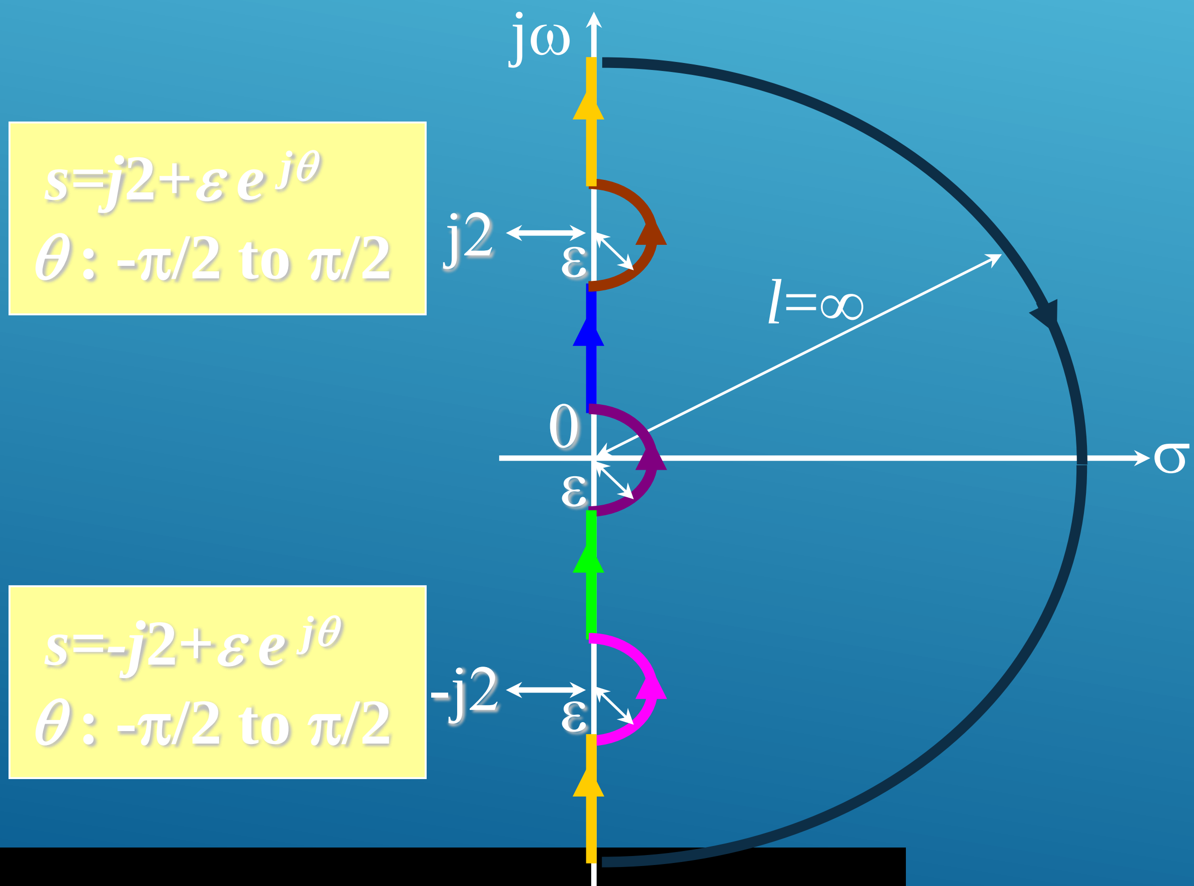
- **G(s)H(s)=1/{s(s+1)}, clearly G(jw)H(jw)=1/{jw(jw+1)}**



Infinitesimal semicircle of radius ε from the s-plane is mapped as an infinity radius semicircle in GH-plane

Frequency Response Analysis
Nyquist Stability Criterion

- **$G(s)H(s)=1/\{s(s^2+4)\}$, clearly $G(jw)H(jw)=1/\{jw(-w^2+4)\}$**



Detour around every jw axis pole by adding a semicircle of infinitesimal radius ε!

The rest is the same...

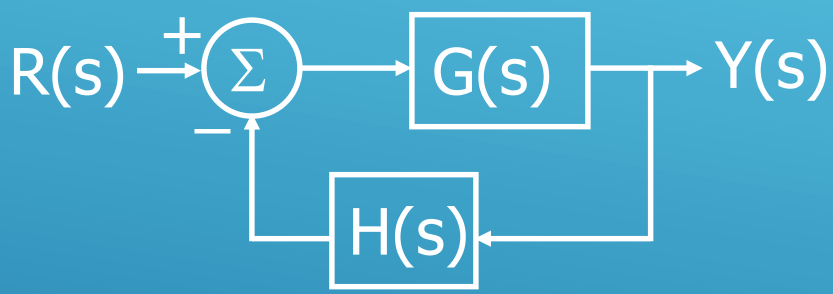
[redacted]

Frequency Response Analysis
Nyquist Stability Criterion

- Choose the clockwise contour in s-plane, such that the right half s-plane is contained entirely.
- Calculate G(jw)H(jw) along this contour. Consider critical points first and choose some intermediate points. Use of a computer may be inevitable...
- Construct the corresponding curve in GH-plane. Pay attention to the rotation (clockwise or counterclockwise).

Now we are ready to give Nyquist Stability Criterion

Frequency Response Analysis
Nyquist Stability Criterion



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

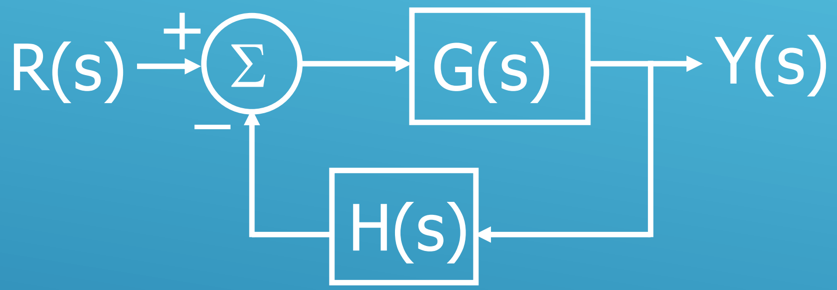
1+G(s)H(s)=0 is the characteristic equation. Nyquist stability criterion lets us know

The number of right half s-plane poles of T(s) by using

The number of right half s-plane poles of G(s)H(s) and

The number of clockwise encirclements of the point -1+j0 made by the polar plot of G(jw)H(jw).

Frequency Response Analysis
Nyquist Stability Criterion



$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

1+G(s)H(s)=0 is the characteristic equation. Nyquist stability criterion lets us know

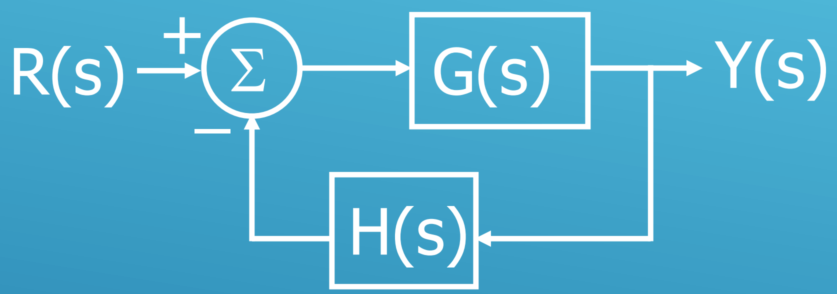
The number of right half s-plane poles of T(s)=Z

The number of right half s-plane poles of G(s)H(s)=P

The number of clockwise encirclements of the point -1+j0 made by the polar plot of G(jw)H(jw)=N

$$Z=N+P$$

Frequency Response Analysis
Nyquist Stability Criterion



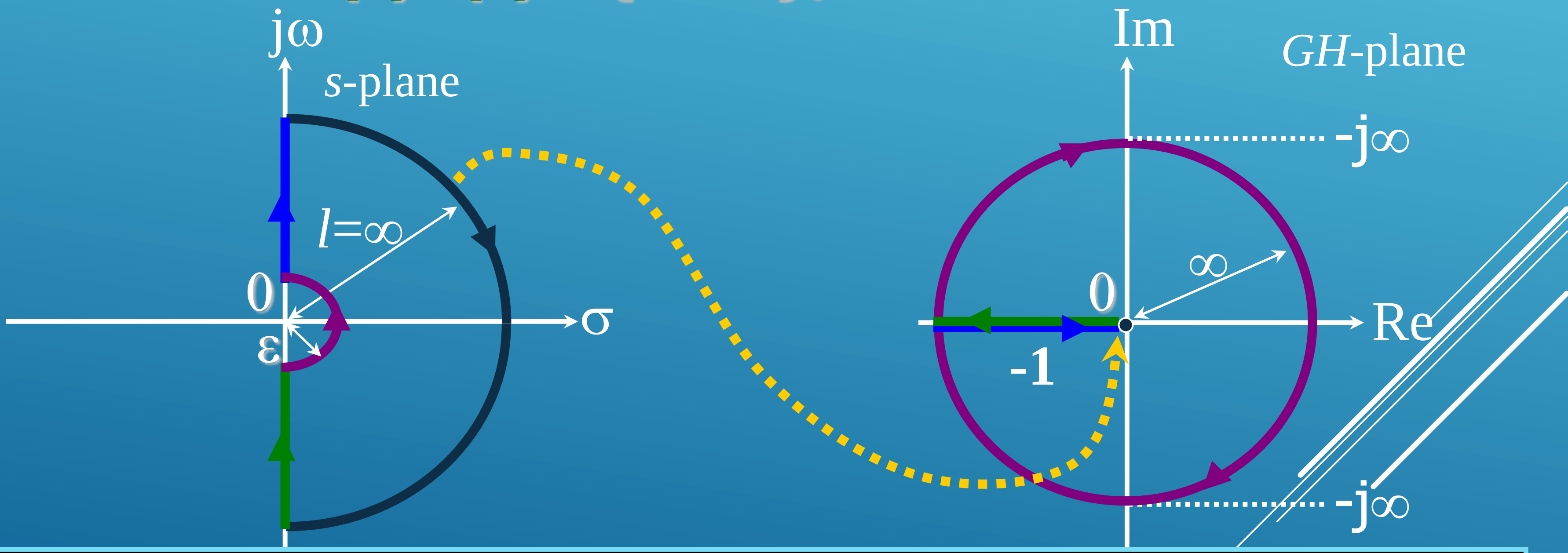
$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

Stable closed loop means Z=0. Obviously this means N=-P
The number of right half s-plane poles of G(s)H(s) must be equal to the number of <u>counterclockwise</u> encirclements of the point -1+j0.

Z=N+P

Frequency Response Analysis
Nyquist Stability Criterion
Example-I

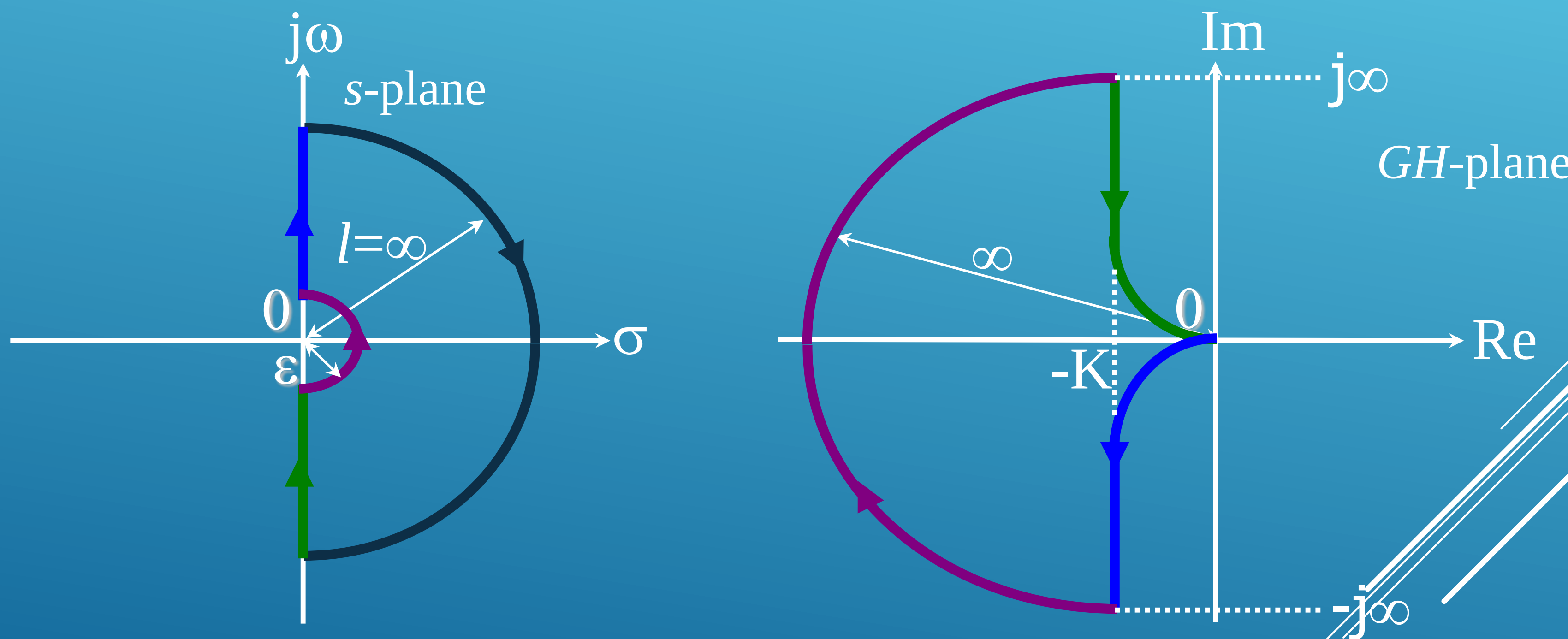
- $G(s)H(s)=K/s^2$, clearly
  $G(jw)H(jw)=-K/w^2$
  $1+G(s)H(s)= \{s^2+K\}/s^2$



Locus passes through -1+j0 point, i.e. the closed loop poles are located on the jw axis, $s^2+K=0$!

Frequency Response Analysis
Nyquist Stability Criterion
Example-II

- **G(s)H(s)=K/s(s-1), clearly P=1**



No matter what K is, locus encircles -1+j0 point one times in the clockwise direction, so N=1

Frequency Response Analysis
Nyquist Stability Criterion
Example-II

- $G(s)H(s)=K/s(s-1)$, $P=1$, $N=1$

  $Z=N+P=1+1=2$

  This result tells us that 2 of the closed loop poles lie on the right half s-plane.

  $$1+G(s)H(s)=\frac{\boxed{s^2-s+K}}{s(s-1)}=0$$

  Zeros of the char. eqn. Have real parts equal to 1/2, i.e. on the right half s-plane.

  You could check the CL stability by using the Routh test as well. See the root locus..

[redacted]

Frequency Response Analysis
Nyquist Stability Criterion

- **If we can use Routh test, why should we use Nyquist stability criterion, which is more time-consuming?**

  Sometimes, you have only the frequency response data of G(s) and/or H(s), which may contain transducers, measurement devices etc. In such cases, Nyquist stability criterion gives a good idea about closed loop stability. Also, the use of Nyquist plots let us see the relative stability properties of the system at hand.

Frequency Response Analysis
Nyquist Stability Criterion

- **What if we have a time delay terms in the open loop transfer function?**

Use the following series expansion, and truncate.

$$e^{-Ts} = \frac{1 - \frac{Ts}{2} + \frac{(Ts)^2}{8} - \frac{(Ts)^3}{48} + \cdots}{1 + \frac{Ts}{2} + \frac{(Ts)^2}{8} + \frac{(Ts)^3}{48} + \cdots}$$

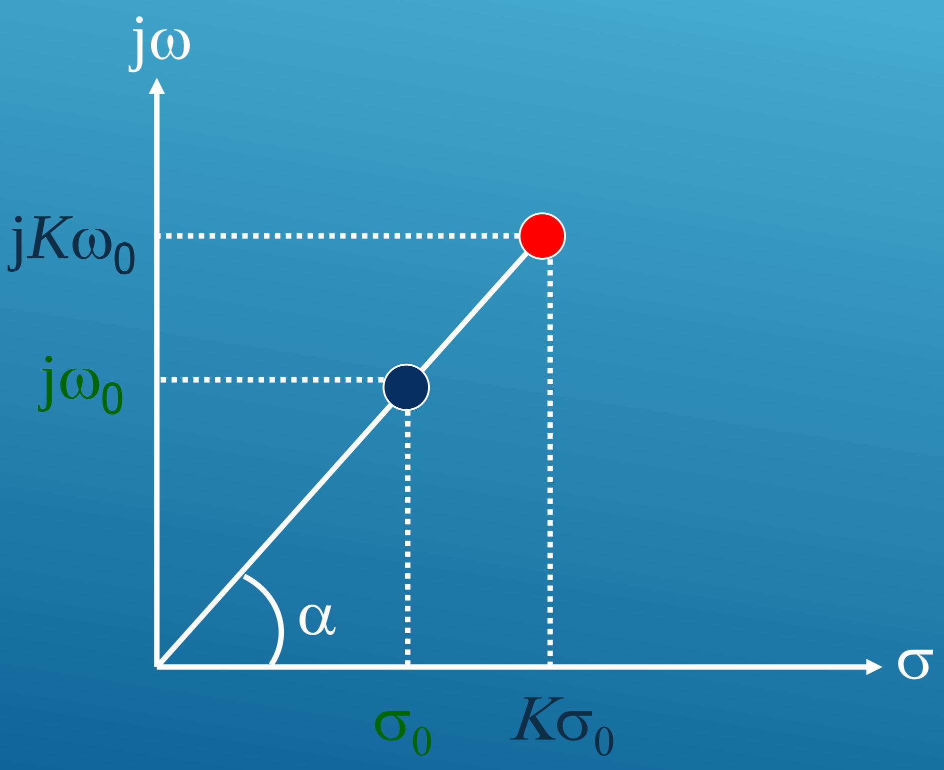
How reliable is this?

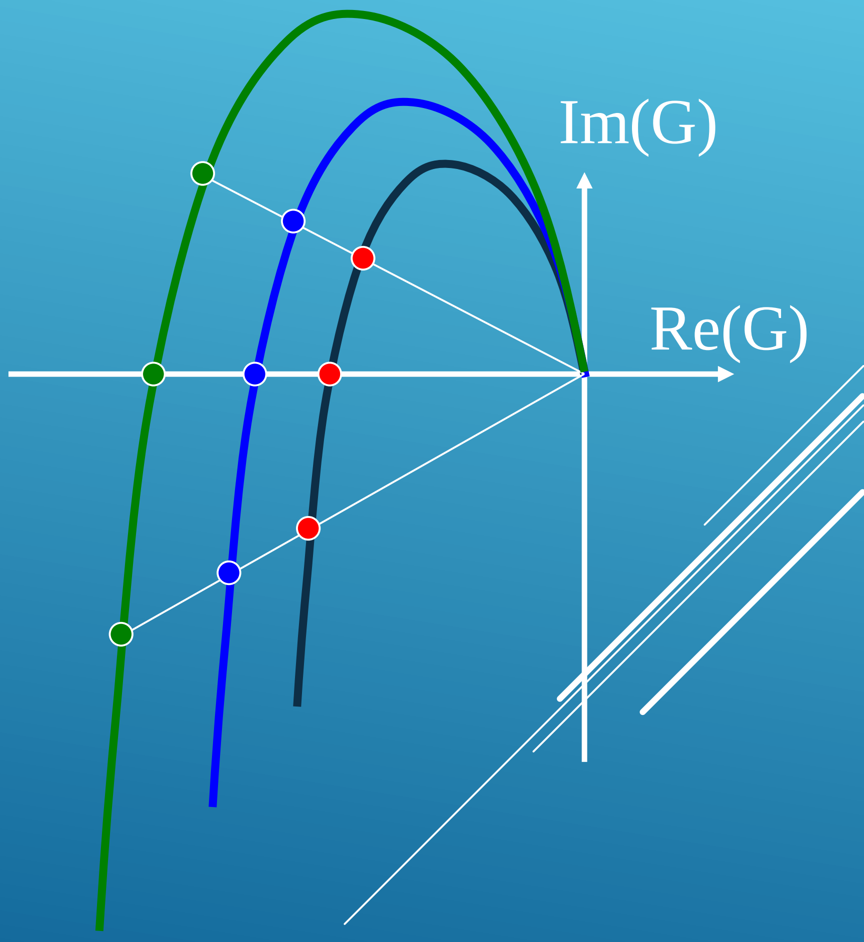
Frequency Response Analysis
Nyquist Stability Criterion

#of unstable Open Loop Poles	#of Clockwise Encirclements of -1+j0 (P)	$of Unstable Closed Loop poles N=Z+P	Feedback System is Stable/Unstable	#of Clockwise Encirclements of -1+j0 (-P)
2	-2	0	Stable	2
0	3	3	Unstable	-3
5	-5	0	Stable	5
0	0	0	Stable	0
1	-1	0	Stable	1
100	-100	0	Stable	100
0	4	4	Unstable	-4

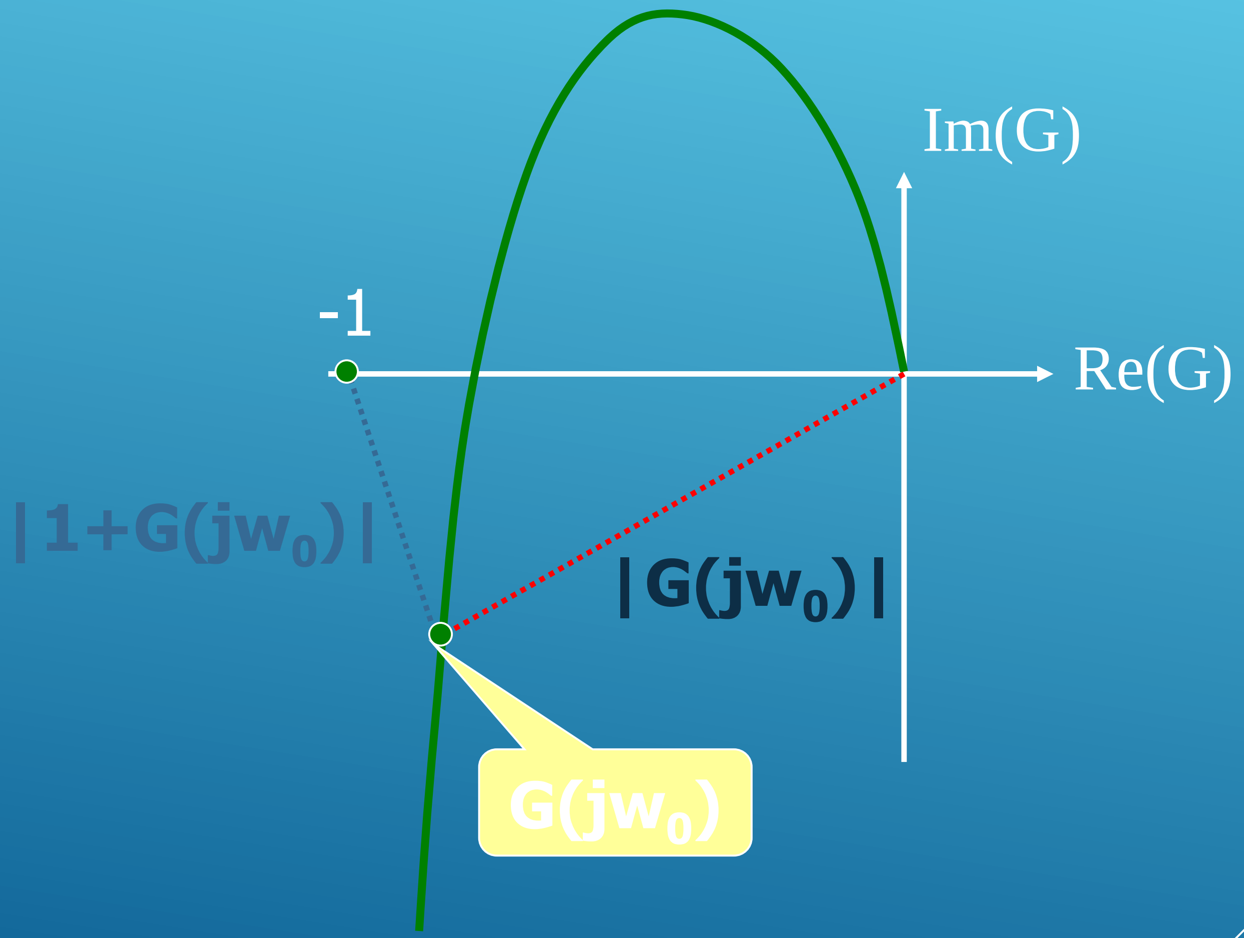
Nyquist Curve is a Conformal Map

- Conformal map is an angle preserving map.

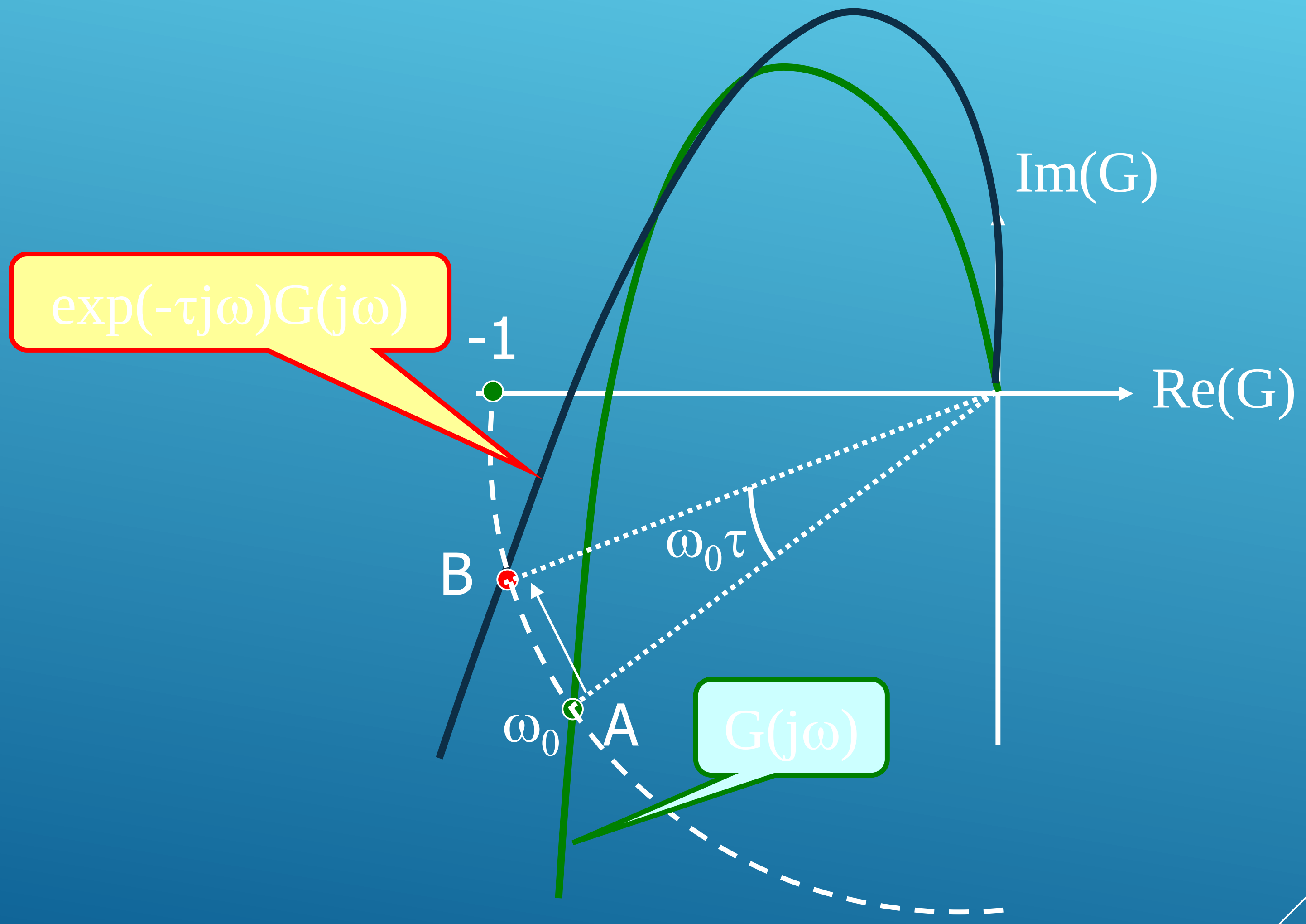




Information on Nyquist Plot
Frequency Response of the Closed Loop



Information on Nyquist Plot
Effect of Delay



Information on Nyquist Plot
Amplification and Attenuation