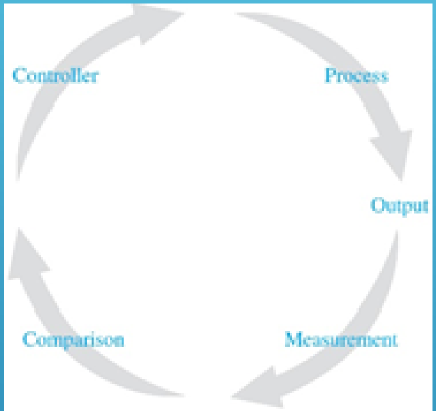
CONTROL SYSTEMS



Doç. Dr. Murat Efe

WEEK 13

This week's agenda

- Design of Control Systems in State Space
  - Canonical Realizations
  - Controllability and Observability
  - Linear State Feedback
  - Pole Placement
  - Bass-Gura and Ackermann Formulations
  - Properties of State Feedback
  - Observer Design and Observer Based Compensators

Canonical Realizations

We will learn

- Controller (or controllability) canonical form
- Observer (or observability) canonical form

Canonical Realizations - Controller C.F.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$y''' + a_1 y'' + a_2 y' + a_3 y = b_1 u'' + b_2 u' + b_3 u$$

Given a strictly proper transfer function, you can write the **differential equation** that describes it. Let $\xi(t)$ be a solution of $y(t)''' + a_1 y(t)'' + a_2 y(t)' + a_3 y(t) = u(t)$

Then the overall solution can be written as

$$y = b_1 \xi'' + b_2 \xi' + b_3 \xi$$

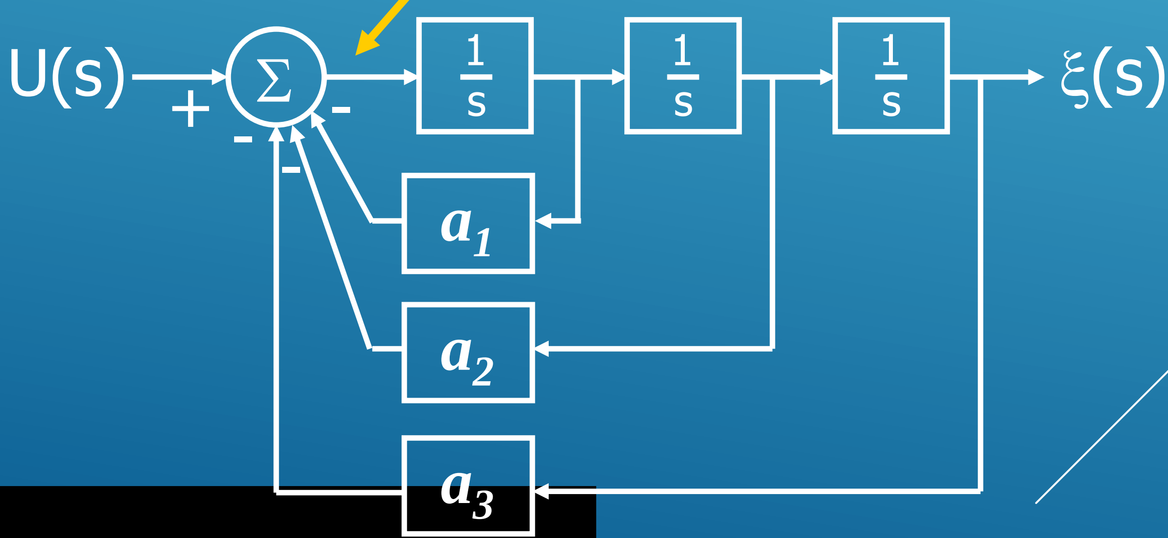
Canonical Realizations - Controller C.F.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$y''' + a_1 y'' + a_2 y' + a_3 y = b_1 u'' + b_2 u' + b_3 u$$

Let's first realize $\xi''' + a_1 \xi'' + a_2 \xi' + a_3 \xi = u$

Or equivalently $\xi''' = u - a_1 \xi'' - a_2 \xi' - a_3 \xi$



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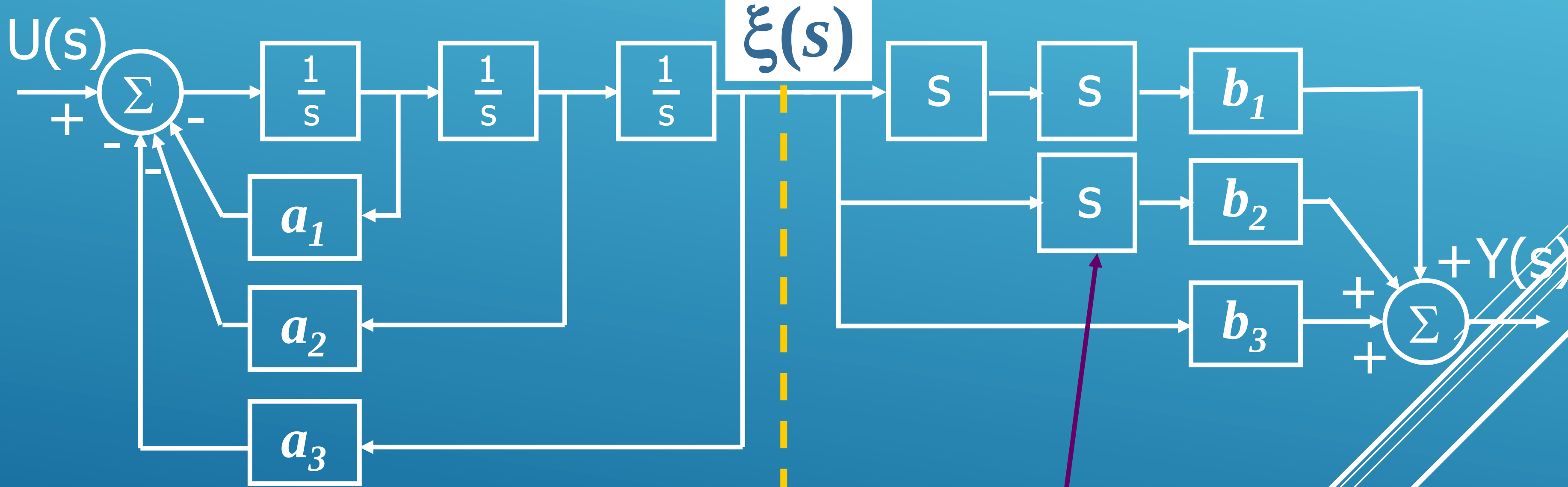
Canonical Realizations - Controller C.F.

Denominator

$$y''' + a_1 y'' + a_2 y' + a_3 y = u$$

Numerator

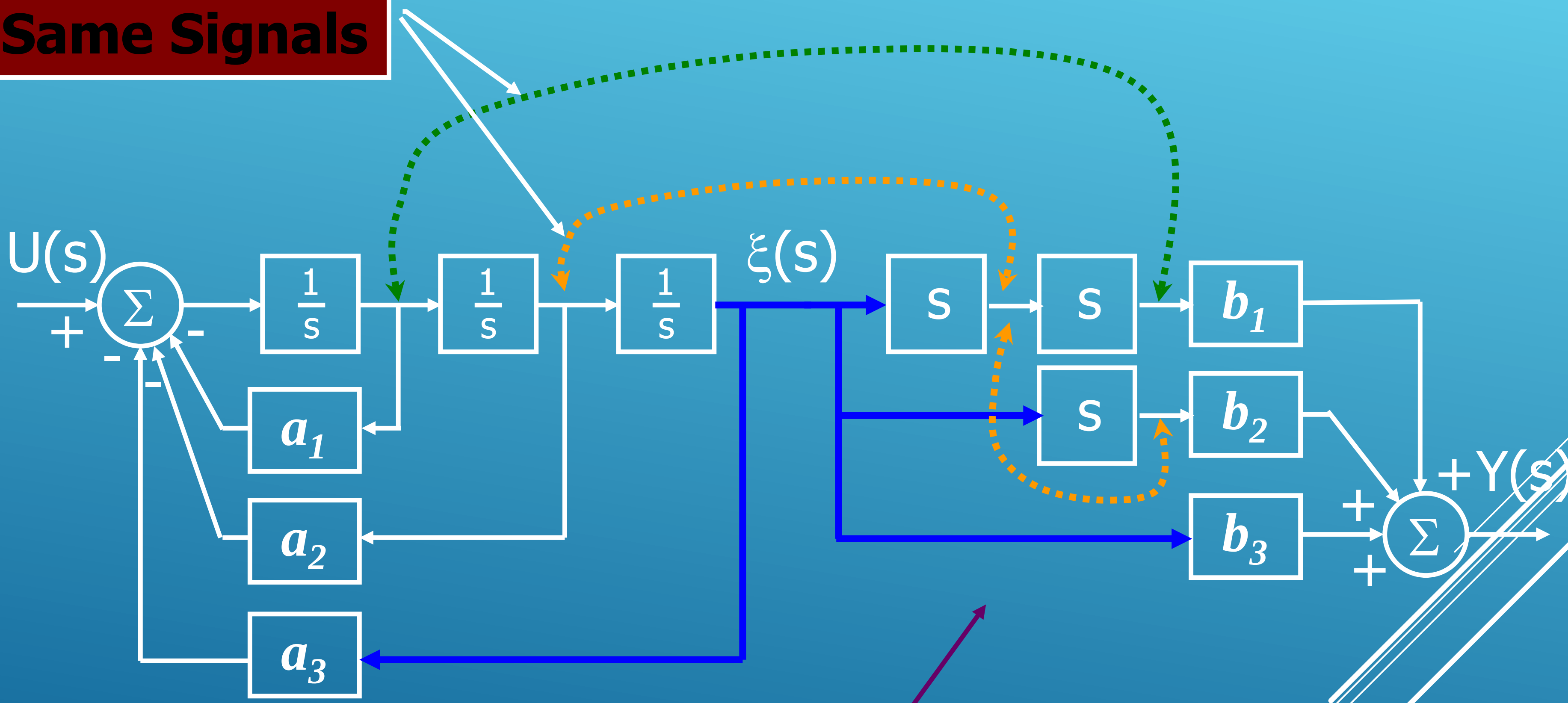
$$y = b_1 \xi'' + b_2 \xi' + b_3 \xi$$



But we do not want differentiators in our realization

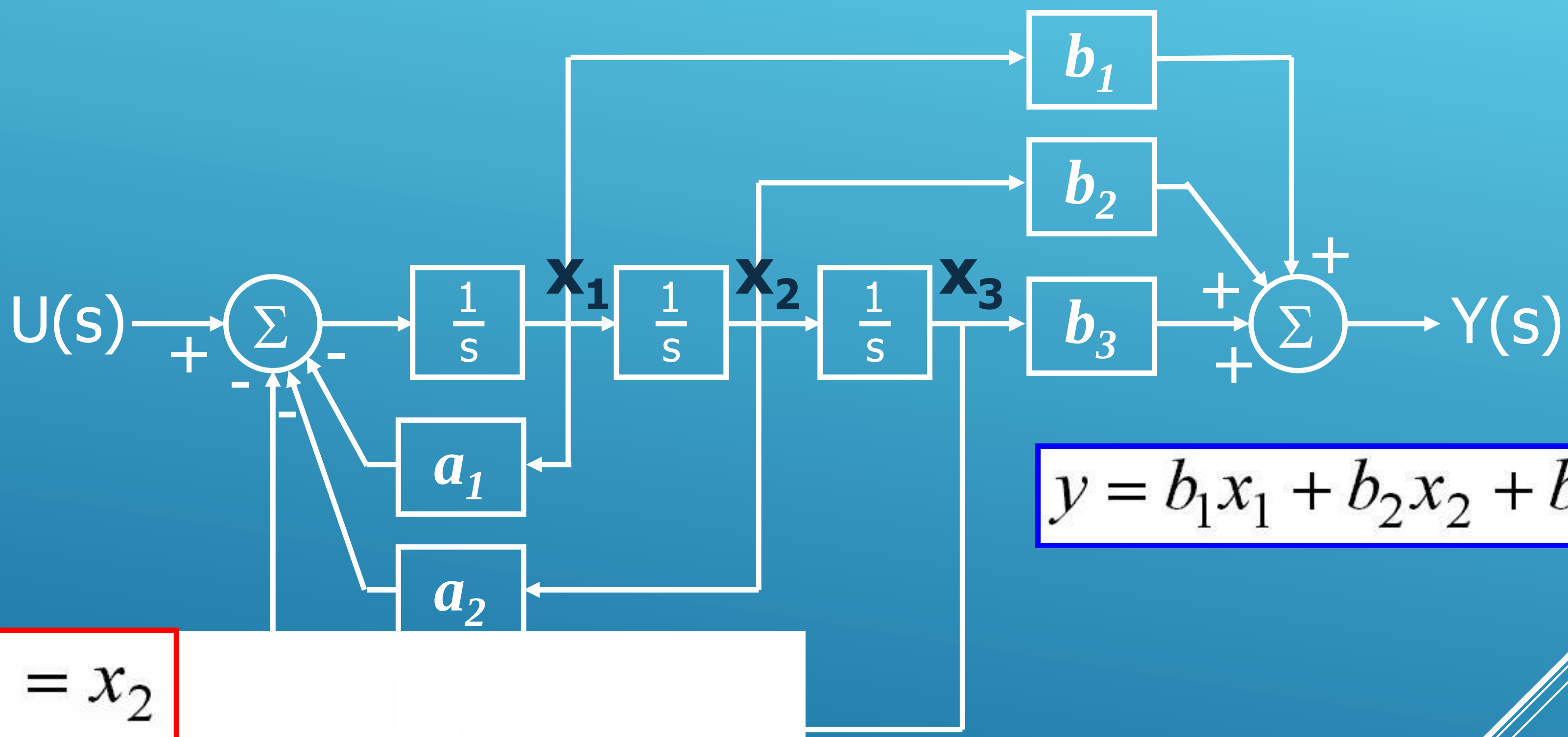
Canonical Realizations - Controller C.F.

Same Signals



So we can rearrange it...

Canonical Realizations - Controller C.F.



$$y = b_1 x_1 + b_2 x_2 + b_3 x_3$$

$$\dot{x}_3 = x_2$$

$$\dot{x}_2 = x_1$$

$$\dot{x}_1 = -a_1 x_1 - a_2 x_2 - a_3 x_3 + u$$

Canonical Realizations - Controller C.F.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x} = Ax + bu$$
$$y = Cx$$

Denote this by (A_c, b_c, C_c)

Note that if the transfer function is not strictly proper, you can always perform the division and obtain a strictly proper transfer function.

Canonical Realizations - Observer C.F.

$$T(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$Y(s) = \frac{1}{a(s)} M(\mathrm{s}) \text{ where } M(s) = b(s)U(s)$$

$$\left(s^3 + a_1 s^2 + a_2 s + a_3\right) Y(s) = M(s)$$

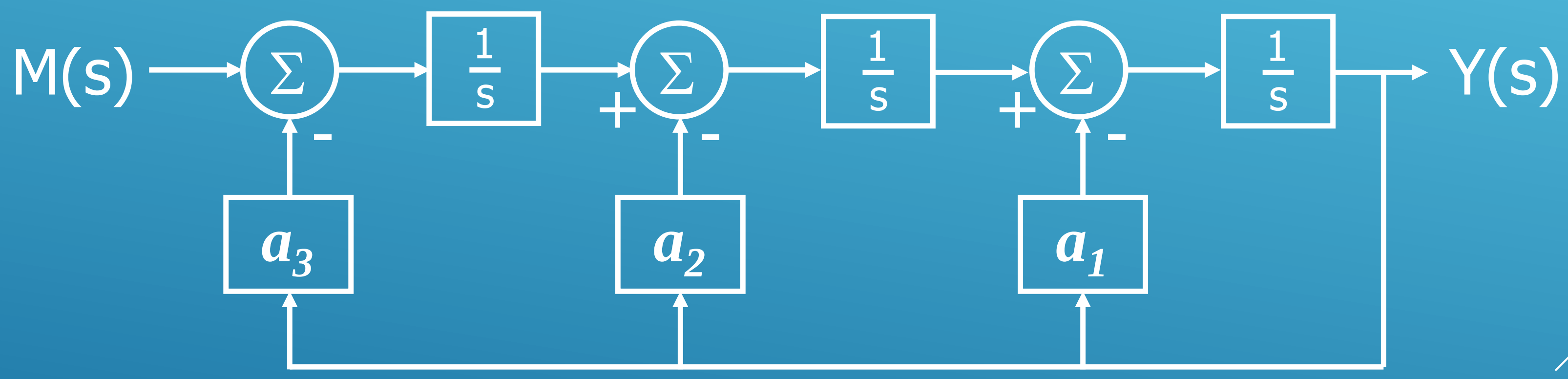
$$s^3 Y(s) = M(s) - a_1 s^2 Y(s) - a_2 s Y(s) - a_3 Y(s)$$

[redacted] $$Y(s) = s^{-3} M(s) - a_1 s^{-1} Y(s) - a_2 s^{-2} Y(s) - a_3 s^{-3} Y(s)$$

$$Y(s) = s^{-1}\left\{- a_1 Y(s) + s^{-1}\left\{- a_2 Y(s) + s^{-1}\left\{- a_3 Y(s) + M(s)\right\}\right\}\right\}$$

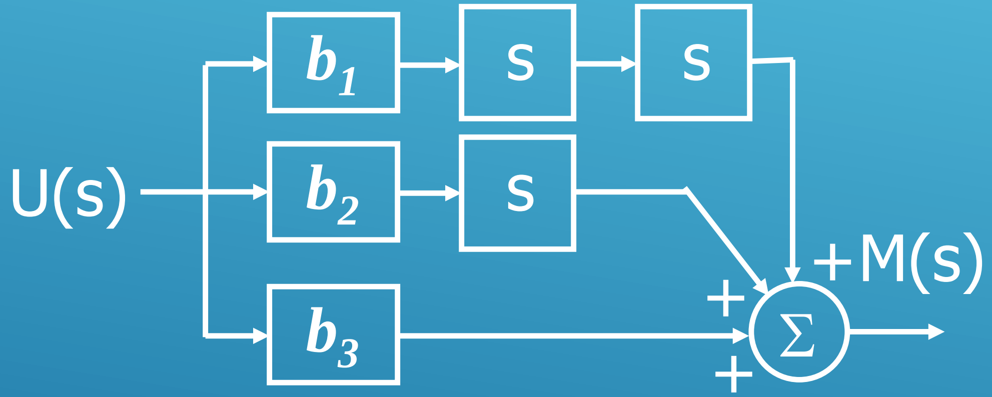
Canonical Realizations - Observer C.F.

$$Y(s) = s^{-1}\left\{- a_1 Y(s) + s^{-1}\left\{- a_2 Y(s) + s^{-1}\{- a_3 Y(s) + M(s)\}\right\}\right\}$$



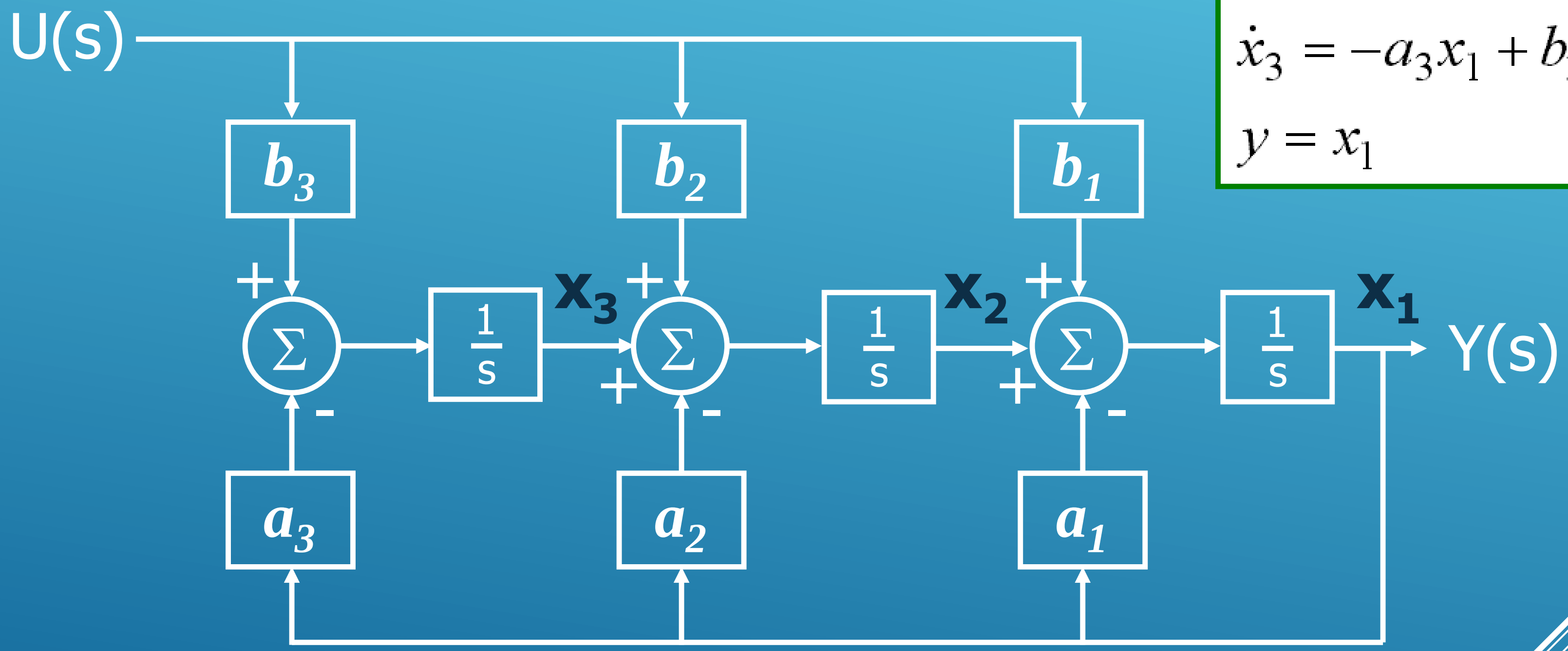
Canonical Realizations - Observer C.F.

$$M(s) = \left(b_1 s^2 + b_2 s + b_3\right) U(s)$$



Combining the two parts and removing the differentiators through seeing the simplifications in the diagram would let us have the following compact representation...

Canonical Realizations - Observer C.F.



$$\dot{x}_1 = -a_1 x_1 + x_2 + b_1 u$$
$$\dot{x}_2 = -a_2 x_1 + x_3 + b_2 u$$
$$\dot{x}_3 = -a_3 x_1 + b_3 u$$
$$y = x_1$$

Canonical Realizations - Observer C.F.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{x} = Ax + bu$$
$$y = Cx$$

Denote this by (A_o, b_o, C_o)

Note that if the transfer function is not strictly proper, you can always perform the division and obtain a strictly proper transfer function.

Canonical Realizations

Controller C.Form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observer C.Form

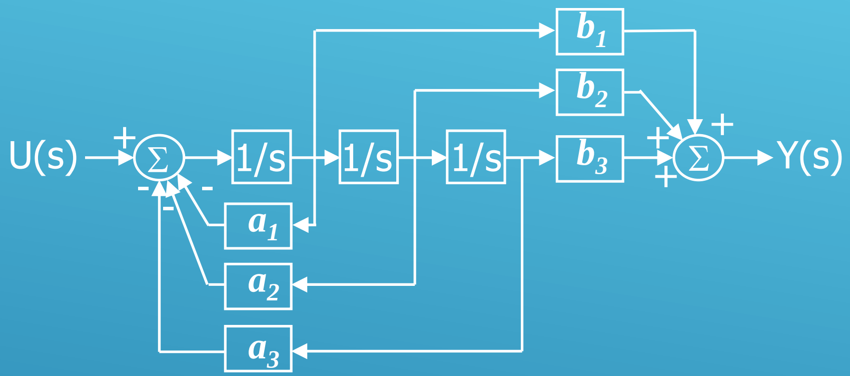
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

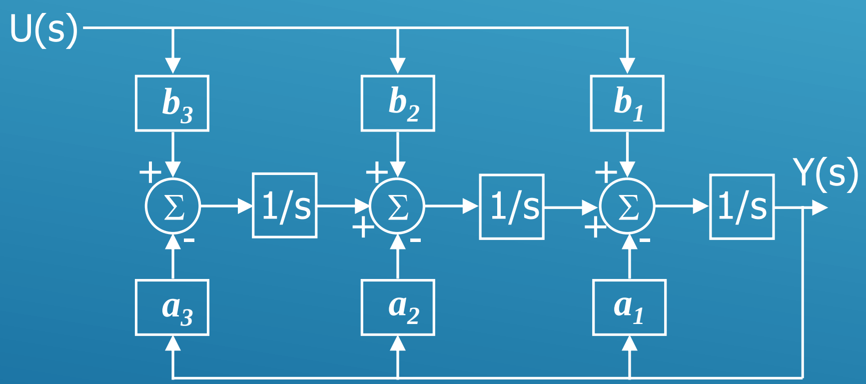
Notice the duality between the controller form realization and observer form realization of a transfer function!

$A_c = A_o^T$
$b_c = C_o^T$
$C_c = b_o^T$

Canonical Realizations - Remarks



Note that in the controller form realization, input affects each x_i either directly or after some integrators. Not every x_i affects the output. Whether or not this will be the case depends on b_i's.



In the observer canonical form, every x_i either directly or after some integrators affects the output. The input, on the other hand, does not have to affect each x_i. Whether or not it does depends again on b_i's.

Hence, controller form realization is not necessarily observable, and observer form realization is not necessarily controllable!

Notes on Realizations

1 Given T(s), we have seen that there are nonunique ways of choosing the internal variables (states). Thus, realizations of T(s) are not unique.

2 If T(s)=b(s)/a(s), then we have seen that there exists a realization of order n=deg a(s).
Note: Order of a realization (A,b,C,d) is the number of internal variables associated with it.

Notes on Realizations

3 If there are simplifications, i.e. the numerator and the denominator are not coprime, you can still realize the transfer function.

$$T(s) = \frac{b(s)}{a(s)} = \frac{\bar{b}(s)(s+\alpha)}{\bar{a}(s)(s+\alpha)} = \frac{\bar{b}(s)}{\bar{a}(s)}$$

n^{th} order realizations

$\bar{n}^{th}$ order realizations

All lead to T(s) but $\bar{n}<n$. Notice that transfer function representation might cancel some important dynamical information!

Notes on Realizations

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

4 Let (A,b,C,d) be a realization of T(s)

P is a nonsingular matrix. Apply the transformation given as

$$\xi(t) = P^{-1}x(t)$$

Calculating the derivative yields

$$\dot{\xi} = P^{-1}\left(Ax + bu\right)$$

Rearrangement gives the new realization

$$\dot{\xi} = P^{-1}AP\xi + P^{-1}bu$$
$$y = CP\xi + du$$

$$(P^{-1}AP, P^{-1}b, CP, d)$$

Notes on Realizations

Does it realize the same TF?

$$\dot{\xi} = P^{-1}AP\xi + P^{-1}bu$$
$$y = CP\xi + du$$

$$T(s) = CP\left(sI - P^{-1}AP\right)^{-1}P^{-1}b + d$$
$$= CP\left(sP^{-1}P - P^{-1}AP\right)^{-1}P^{-1}b + d$$
$$= CP\left(P^{-1}(sP - AP)\right)^{-1}P^{-1}b + d$$
$$= CP\left(P^{-1}(sI - A)P\right)^{-1}P^{-1}b + d$$
$$= CPP^{-1}(sI - A)^{-1}PP^{-1}b + d$$
$$= C(sI - A)^{-1}b + d$$

Yes...

This discussion shows that there may be many different realizations having the same transfer function.

Controllability and Observability

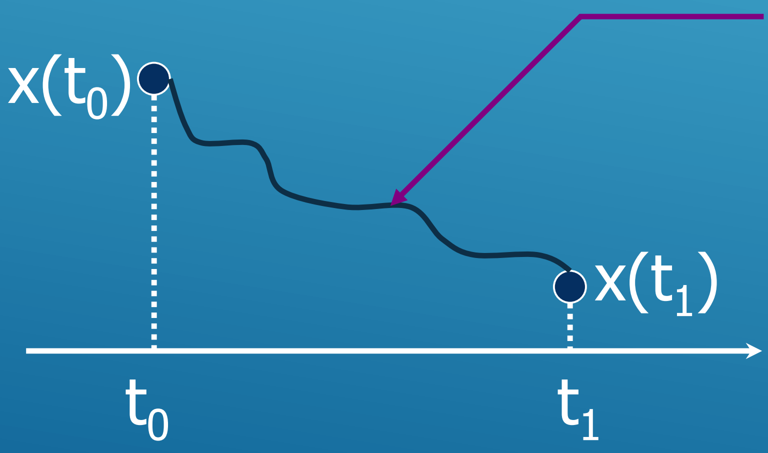
<u>Important note</u>

Controllability and observability are structural properties of the dynamic system.

These issues are NOT the structure or parameters of a control law!

Controllability

- A system is said to be controllable if it is possible by means of an unconstrained control signal to transfer the system from **any** initial state $\underline{x}(t_0)$ to **any** other state $\underline{x}(t_1)$ in **finite** interval of time.



If a control input can lead to this transition, then the system is controllable.
That is to say, the states of your system feels the control input and evolves according to it.

Controllability

Given

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

where A is nxn
b is nx1, C is 1xn and
d is 1x1

Calculate

$$W_c = \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{n-1} b \end{bmatrix}$$

If rank(W_c)=n then the system is said to be complete state controllable.

Controllability

- **Given**

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

where A is nxn
b is nx1, C is 1xn and
d is 1x1

A necessary and sufficient condition for complete state controllability is no cancellation in the following product:

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

If cancellation occurs, the system cannot be controlled in the direction of the canceled mode.

Observability

- A system is said to be observable if **every** state $\underline{x}(t_0)$ can be determined from the observation of **y(t)** over a **finite** time interval $t_0 \le t \le t_1$ (u is available).

Given

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

where A is nxn b is nx1, C is 1xn and d is 1x1

Calculate W_o

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If rank(W_o)=n then the system is said to be completely observable.

Observability

- **Given**

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

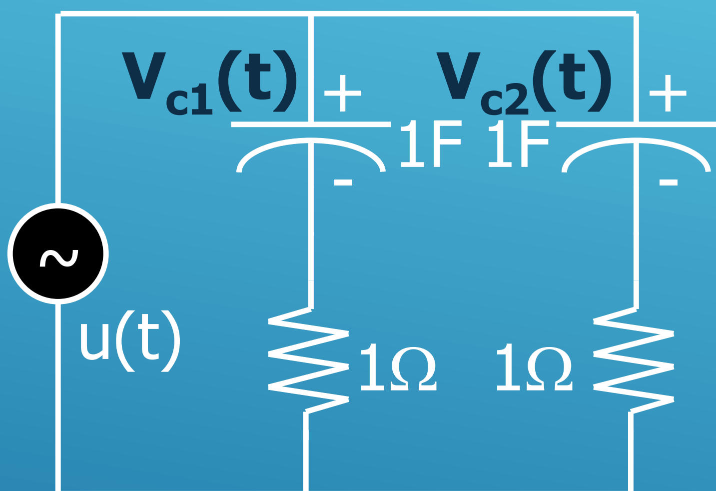
where A is nxn
b is nx1, C is 1xn and
d is 1x1

A necessary and sufficient condition for complete observability is no cancellation in the following product:

$$T(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

If cancellation occurs, then the canceled mode cannot be observed in the output!

Example - I
Check the controllability of the circuit.

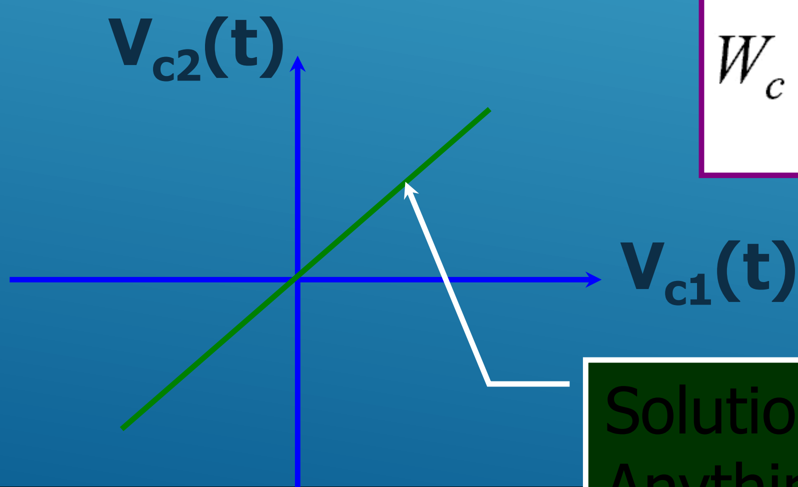


$$x = \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix}, \quad \text{Remember } i_c = C\frac{dVc}{dt}$$

$$u = v_{c1} + \dot{v}_{c1}$$
$$u = v_{c2} + \dot{v}_{c2}$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$W_c = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{rank} W_c = 1$$



Solution is confined to this subspace
Anything outside it cannot be reached!

Example - I
Check the controllability of the circuit.
See the cancellation!

$$T(s) = \begin{bmatrix} \bullet & \bullet \end{bmatrix} \left(sI - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \bullet$$

$$= \begin{bmatrix} \bullet & \bullet \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \bullet$$

$$= \begin{bmatrix} \bullet & \bullet \end{bmatrix} \frac{\begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}}{(s+1)^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \bullet$$

$$= \begin{bmatrix} \bullet & \bullet \end{bmatrix} \frac{(s+1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(s+1)^2} + \bullet$$

One of the modes disappears!

Example - II
Check the observability of the system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad rank\, W_o = 1$$

Apparently not observable…
See the cancellation below

$$T(s) = \frac{(s+2)}{(s+2)(s+1)}$$

Controllability and Observability

Controllability refers to finding an input that drives the states of a dynamical system to any desired position in the state space while observability is to identify the states of the system from input and output measurements.

Minimal Realization Theorem for SISO Case

(A,b,C,d) quadruple is minimal if

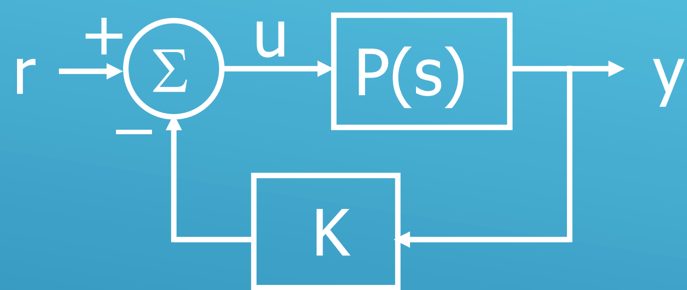
$\Updownarrow$

(A,b) is controllable and (C,A) is observable

$\Updownarrow$

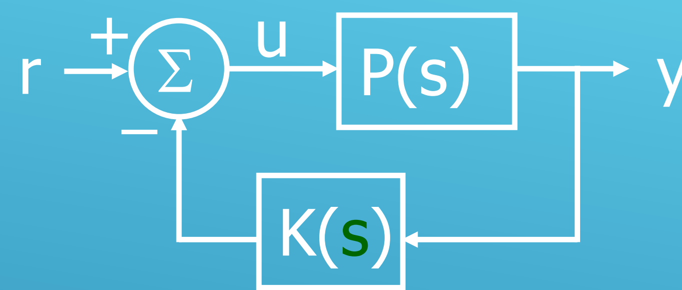
$T(s)=C(sI-A)^{-1}b+d$ is irreducible (no cancellations)

Linear State Feedback
Different types of feedback



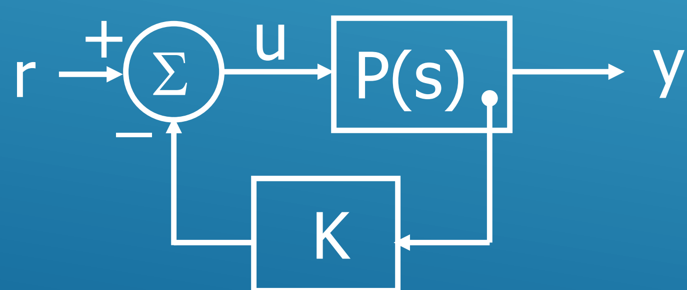
$u=-Ky+r$

Static Output Feedback



$U(s)=-K(s)Y(s)+R(s)$

Dynamic Output Feedback



$u=-Kx+r$

Linear (Constant) State Feedback

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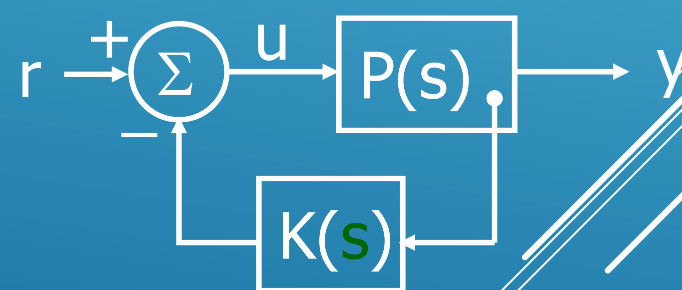


$U(s)=-K(s)X(s)+R(s)$

Dynamic State Feedback

Linear State Feedback

$$\dot{x} = Ax + bu$$
$$y = Cx + du$$

$$u = -Kx + r$$

$$\dot{x} = Ax + b(-Kx + r)$$
$$y = Cx + d(-Kx + r)$$

$$\dot{x} = (A - bK)x + br$$
$$y = (C - dK)x + dr$$

$$P(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

$$T(s) = \frac{Y(s)}{R(s)} = (C - dK)\left(sI - (A - bK)\right)^{-1}b + d$$

How would you choose K such that the closed loop TF meets the desired characteristics?

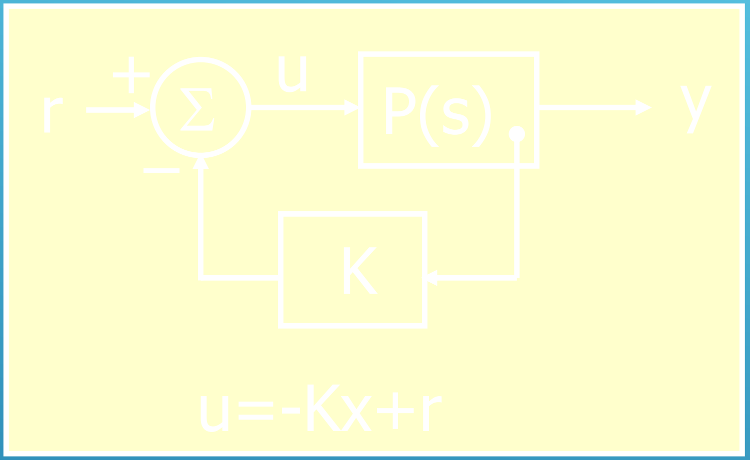
Linear State Feedback

$$P(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}b + d$$

$$T(s) = \frac{Y(s)}{R(s)} = (C - dK)\left(sI - \left(A - bK\right)\right)^{-1}b + d$$

Apparently, the new closed loop poles are now the eigenvalues of the matrix A-bK. If you want to locate the closed loop poles at some desired locations, several methods would let you do this.

Pole Placement
A necessary and sufficient condition



If the pair (A,b) is completely state controllable, then the poles of T(s) can be assigned arbitrarily.

Bass-Gura and Ackermann Formulations for Pole Placement

Characteristic eqn

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$

Desired char. eqn.

$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

Bass-Gura Formula

$$K = [\alpha_1 - a_1 \quad \cdots \quad \alpha_n - a_n] \Omega^{-1} W_c^{-1}$$

where

$$\Omega = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ 0 & 0 & 0 & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Bass-Gura and Ackermann Formulations for Pole Placement

Characteristic eqn

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$

Desired char. eqn.

$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

Ackermann Formula

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times n} W_c^{-1} \alpha(A)$$

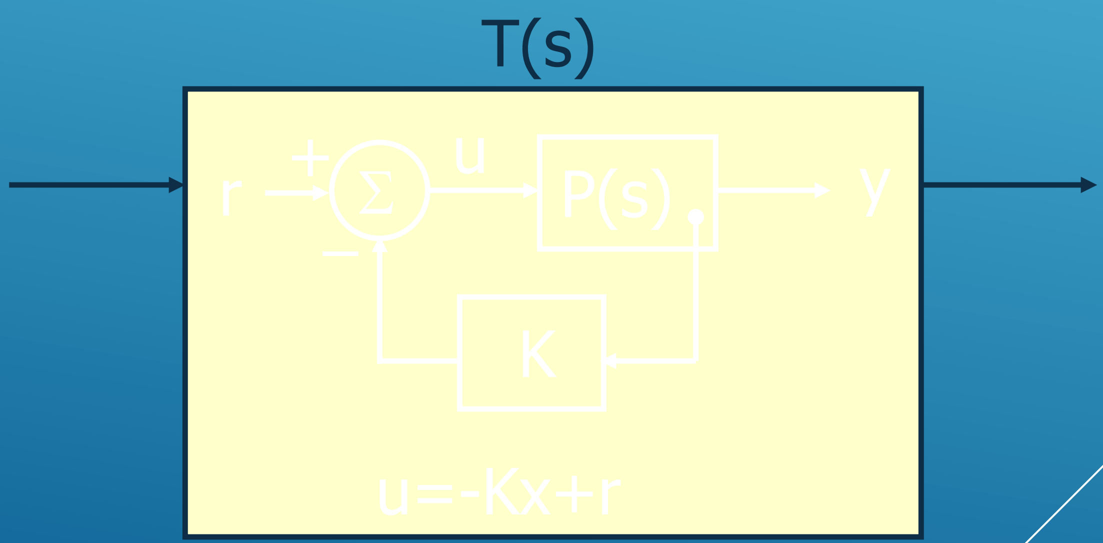
$$\alpha(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

Properties of State Feedback

State Feedback and Zeros

Zeros remain unchanged after state feedback

$$P(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad T(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$



Properties of State Feedback

- **State Feedback and Controllability**

State feedback preserves controllability

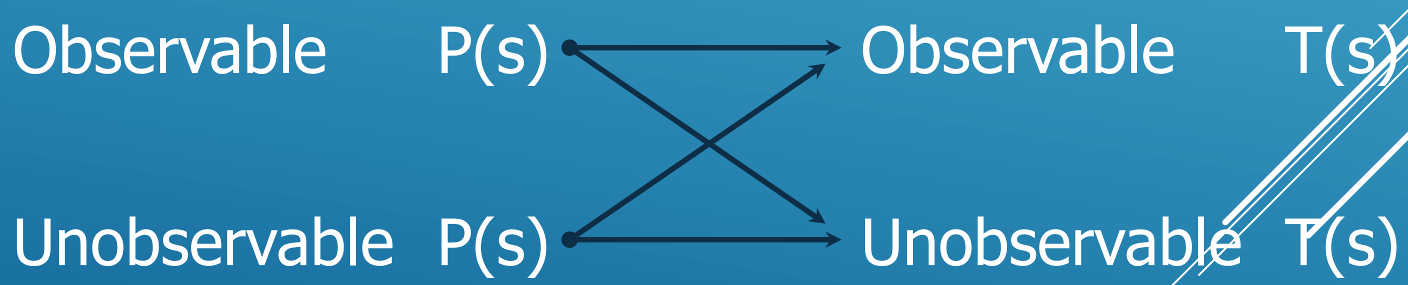
$$P(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad T(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

Properties of State Feedback

State Feedback and Observability

Observability is not necessarily preserved under state feedback.
Neither is the unobservability.

$$P(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad T(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

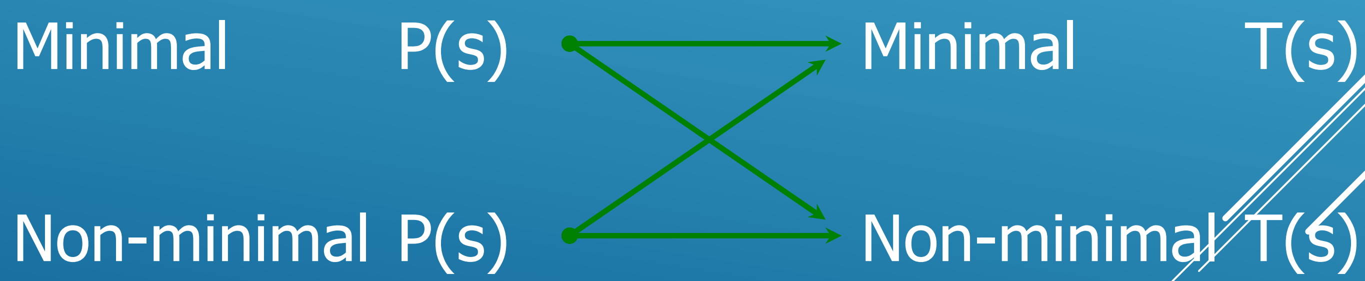


Properties of State Feedback

State Feedback and Minimality

Due to a possible loss of observability, minimality is not necessarily preserved.

$$P(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \qquad T(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$



An Example to State Feedback

Find K

$$\dot{x} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

$$P(s) = \frac{3s^2 + 3s - 6}{s^3 + s^2 - 4s + 2}$$

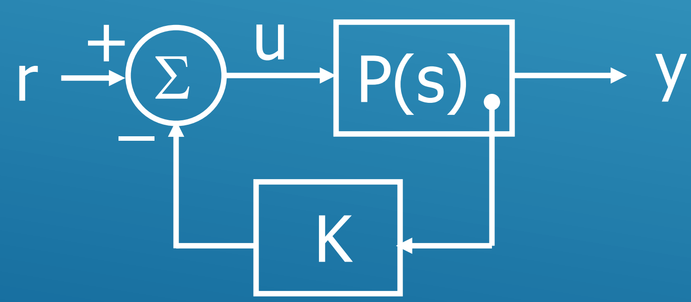
$$|sI - A| = a(s) = s^3 + s^2 - 4s + 2$$

$$T(s) = \frac{3s^2 + 3s - 6}{(s+1)^3}$$

$$|sI - (A - bK)| = \alpha(s) = s^3 + 3s^2 + 3s + 1$$

Desired characteristic equation

T(s)



u=-Kx+r

An Example to State Feedback

```
Transfer function:
 3 s^2 + 3 s - 6
------------------
s^3 + s^2 - 4 s + 2

ans =

  -2.7321
   1.0000
   0.7321

ans =

   3   2

Bass-Gura Formula

K1 =

   0.4211   0.1842   1.0263

ans =

  -1.0000
  -1.0000 + 0.0000i
  -1.0000 - 0.0000i

ans =

   3   3

Ackermann Formula

K2 =

   0.4211   0.1842   1.0263

ans =

  -1.0000
  -1.0000 + 0.0000i
  -1.0000 - 0.0000i

ans =

   3   3
```

```
clear all
close all
clc

A = [1 2 0; 0 -1 3; 0 1 -1];
b = [1 3 1]';
C = [0 1 0];
d = 0;

[numOL,denOL] = ss2tf(A,b,C,d);

h = tf(numOL,denOL)
roots(denOL)

Wc = [b A*b A*A*b];
Wo = [C;C*A;C*A*A];
[rank(Wc) rank(Wo)]

%%%%%%%%%%%%%%%%%%%%%%%%%%
disp(' Bass-Gura Formula')
alpha = [1 3 3 1];
a     = denOL;
Omega = [1 a(2) a(3);0 1 a(2);0 0 1];

K1 = (alpha(2:4)-a(2:4))*inv(Omega)*inv(Wc)
eig(A-b*K1)

Wc1 = [b (A-b*K1)*b (A-b*K1)^2*b];
Wo1 = [C;C*(A-b*K1);C*(A-b*K1)^2];
[rank(Wc1) rank(Wo1)]

%%%%%%%%%%%%%%%%%%%%%%%%%%
disp(' Ackermann Formula')
alpha = [1 3 3 1];
alpha_of_A = zeros(3,3);
for i=1:4
   alpha_of_A = alpha_of_A + alpha(i)*A^(4-i);
end
K2 = [0 0 1]*inv(Wc)*alpha_of_A
eig(A-b*K2)

Wc2 = [b (A-b*K2)*b (A-b*K2)^2*b];
Wo2 = [C;C*(A-b*K2);C*(A-b*K2)^2];
[rank(Wc2) rank(Wo2)]
```

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$

$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

$$K = [\alpha_1 - a_1 \quad \cdots \quad \alpha_n - a_n]\Omega^{-1} W_c^{-1}$$

$$\Omega = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ 0 & 0 & 0 & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$

$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

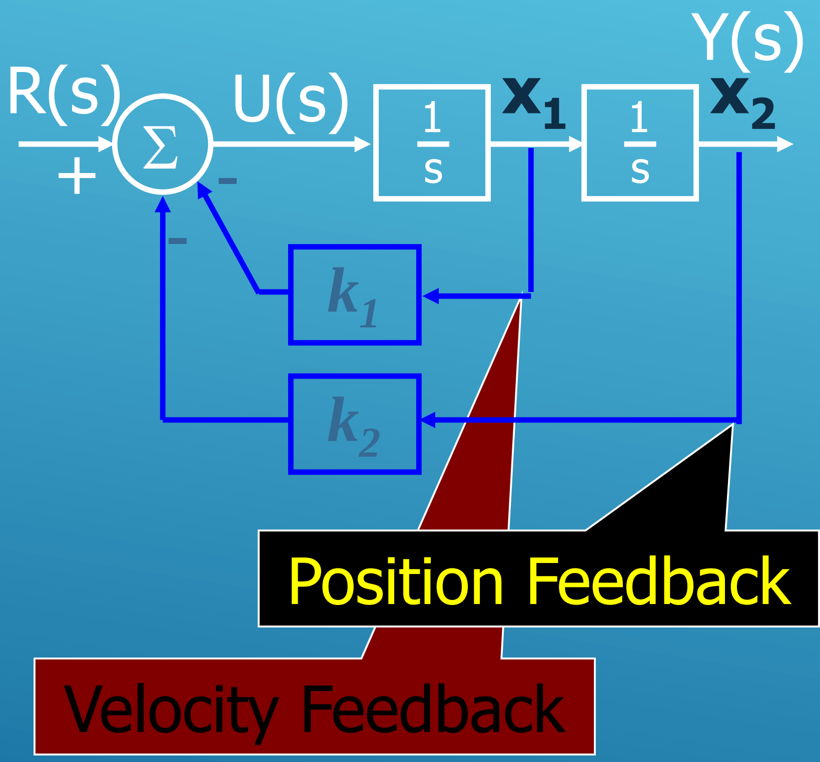
$$K = [0 \quad 0 \quad \cdots \quad 0 \quad 1]_{1\times n} W_c^{-1} \alpha(A)$$

$$\alpha(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

An Example to State Feedback

- **The zeros remain unchanged (Show this by Matlab)**
- **(A,b) is controllable, so is (A-bK,b)**
- **(C,A) is unobservable, but (C,A-bK) is observable**
- **Notice that you arrived at the same K with both Bass-Gura and Ackermann formulas**

An Example
Double Integrator



Controllability canonical form

$$\left.\begin{aligned}\dot{x}_1 &= u\\ \dot{x}_2 &= x_1\\ y &= x_2\end{aligned}\right\}\quad \frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$W_c = \begin{bmatrix}b & Ab\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \quad \text{rank } W_c = 2$$

$$W_o = \begin{bmatrix}C\\CA\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \quad \text{rank } W_o = 2$$

$$u = -Kx + r = -k_1x_1 - k_2x_2 + r$$

$$u = -\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + r$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}r$$

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-k_1 & -k_2\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}r$$

$$T(s) = \frac{1}{s^2 + k_1s + k_2}$$

Matlab Shortcuts

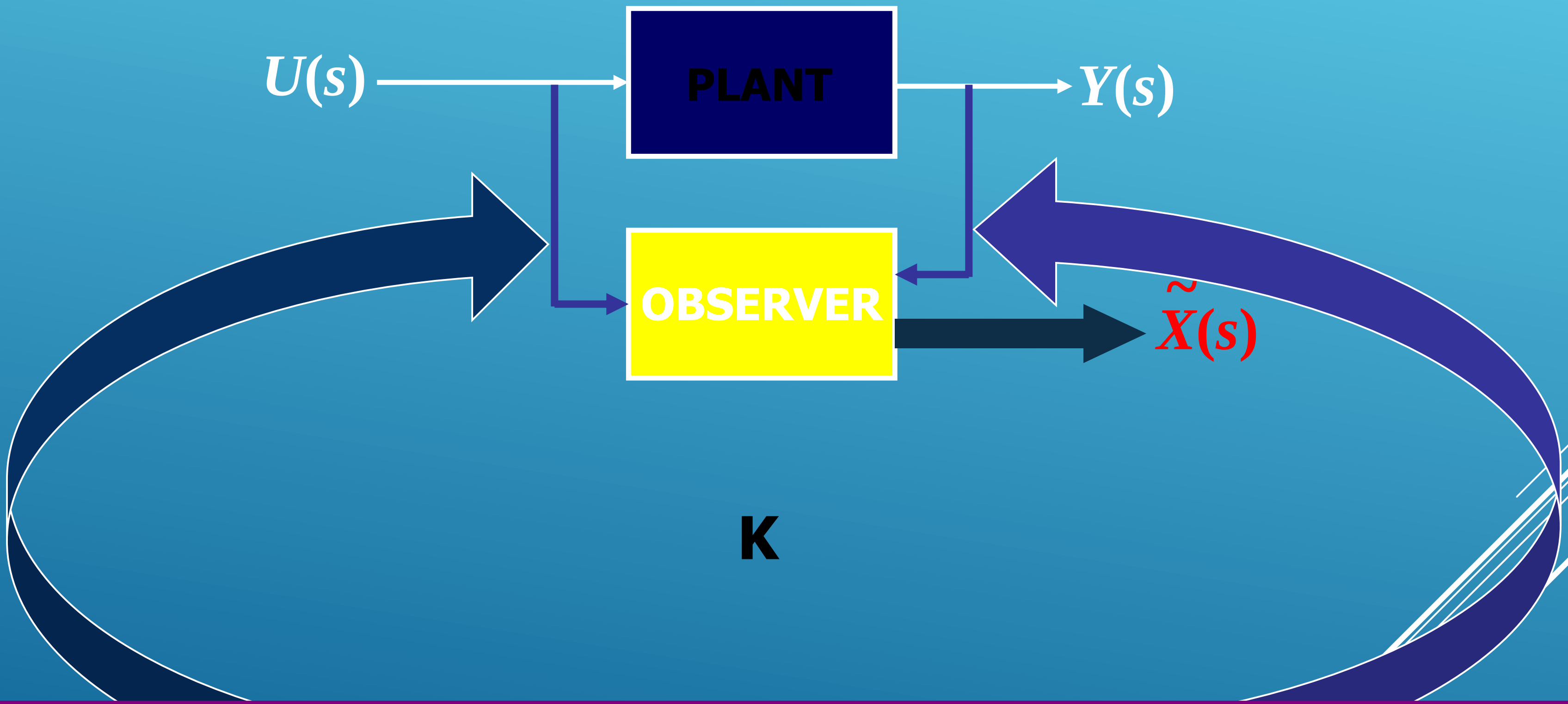
- Type »**help place** for Bass-Gura formula
- Type »**help acker** for Ackermann formula

These will let you know the specifications and algorithmic limitations in Matlab.

A Remark on State Feedback

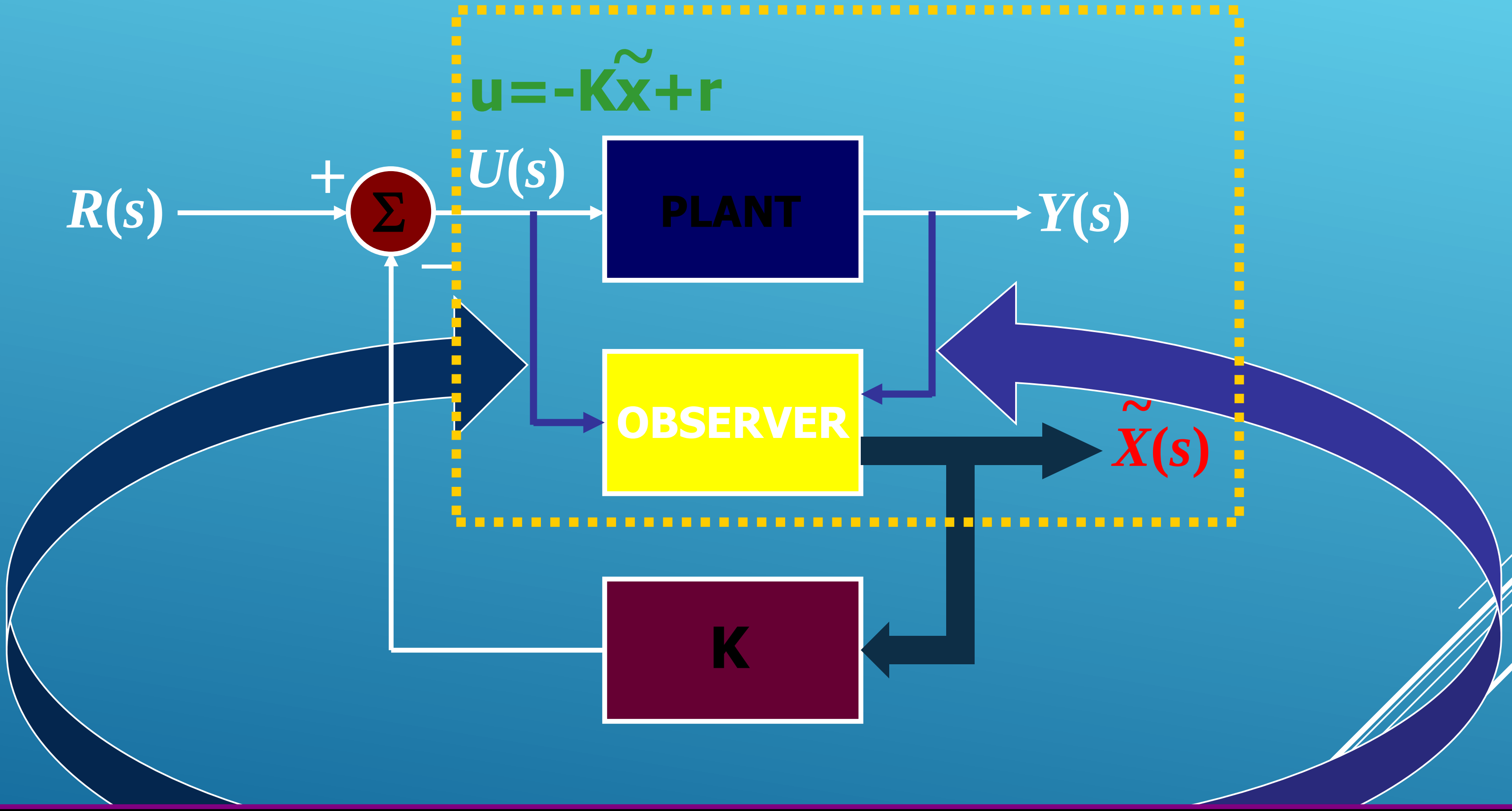
- **In some applications, not all of the states are available for feedback, and we do not want to use differentiators to generate one state from another. In such cases, we need to use other techniques to generate unmeasurable state variables.**

Observer Design and Observer Based Compensators



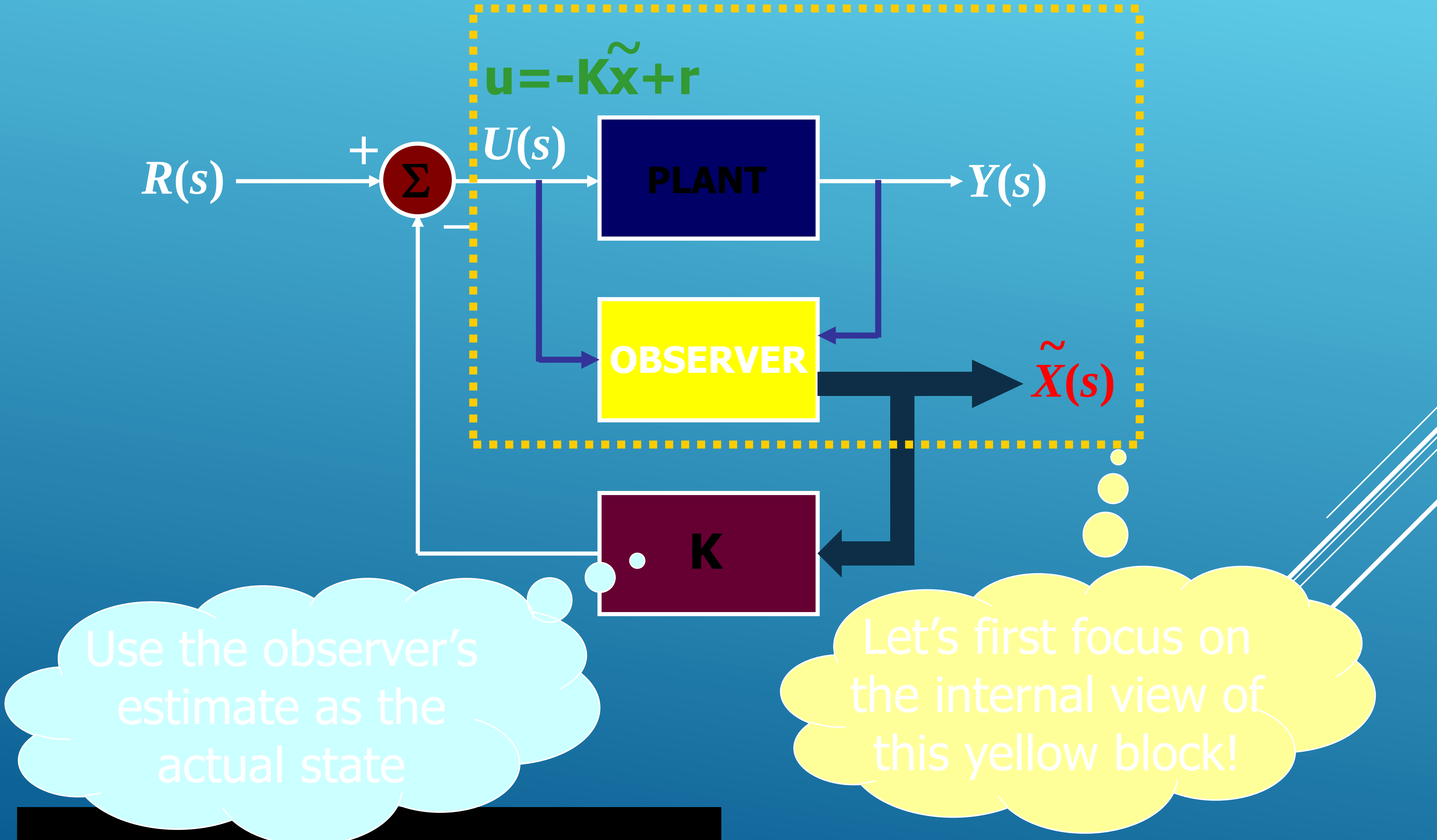
A state observer estimates the state variables based on the measurements of output and control variables.

Observer Design and Observer Based Compensators



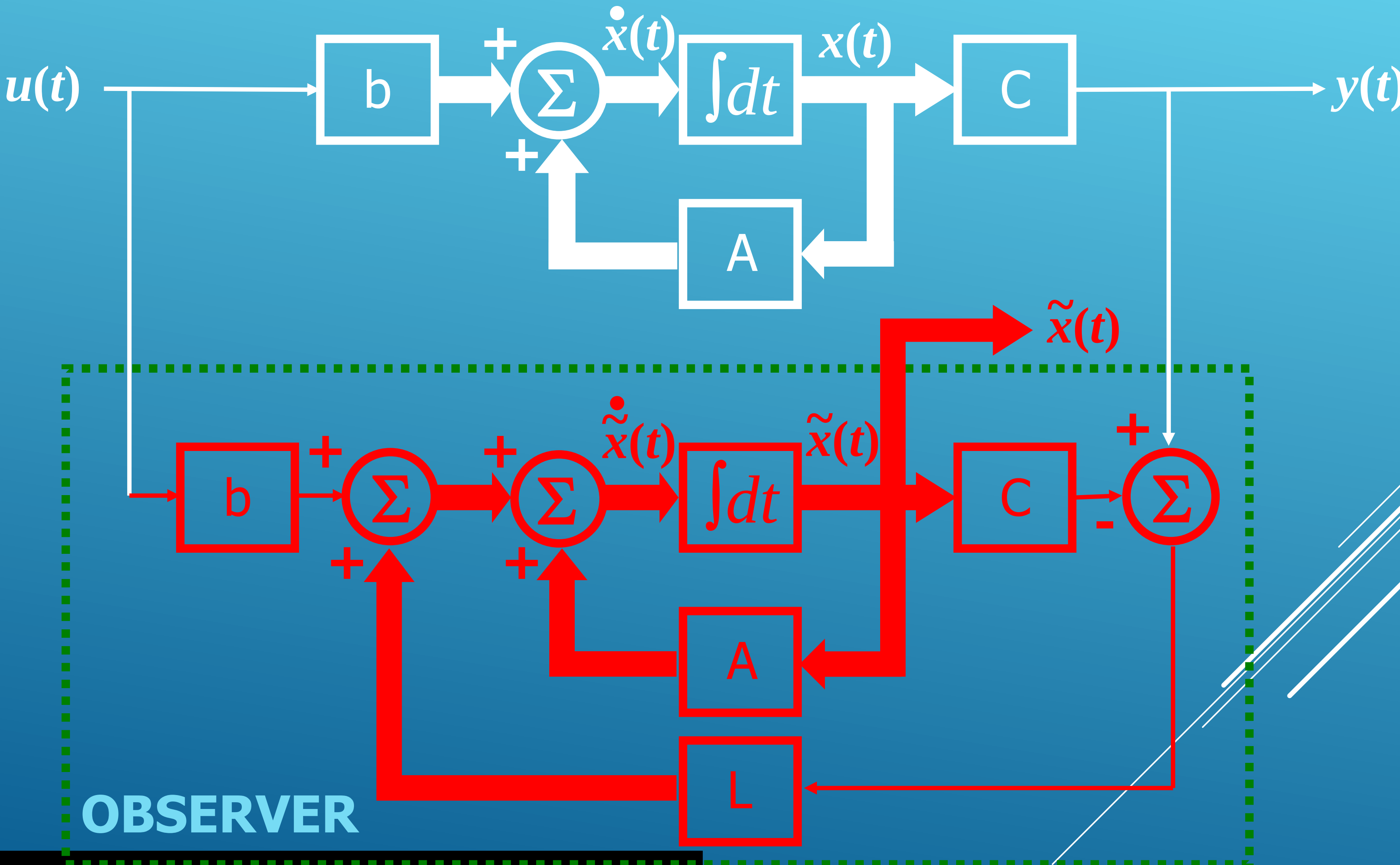
A state observer estimates the state variables based on the measurements of output and control variables.

Observer Design and Observer Based Compensators



[redacted]

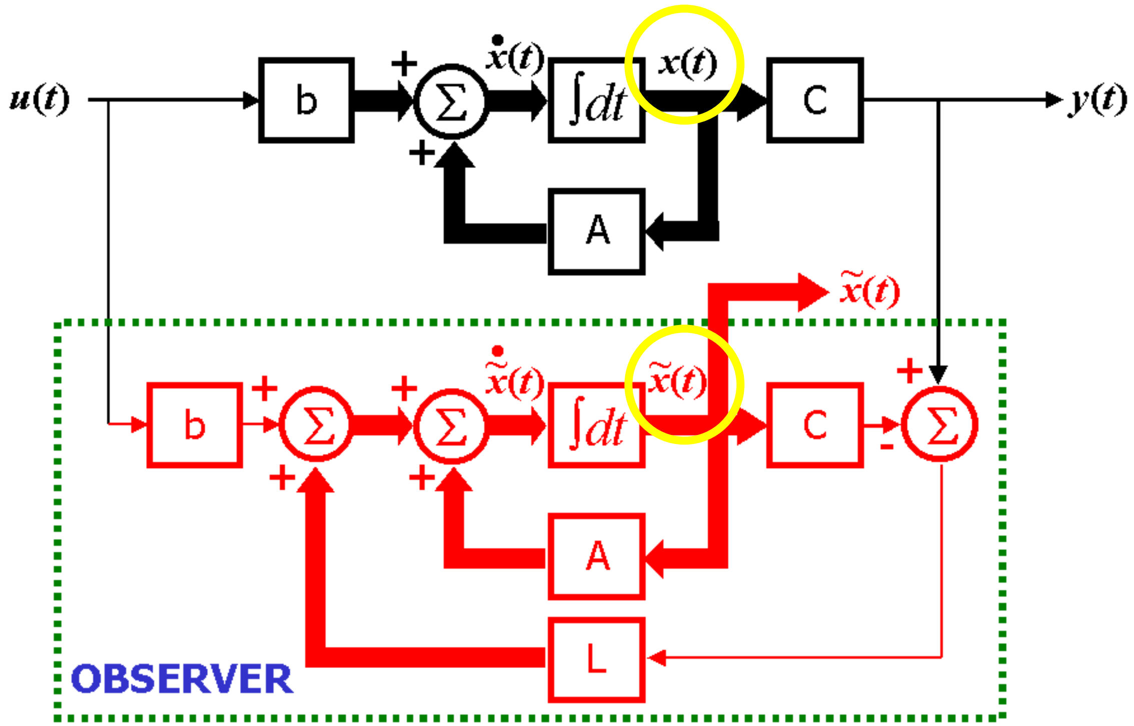
Observer Design and Observer Based Compensators



[redacted]

Observer Design and Observer Based Compensators

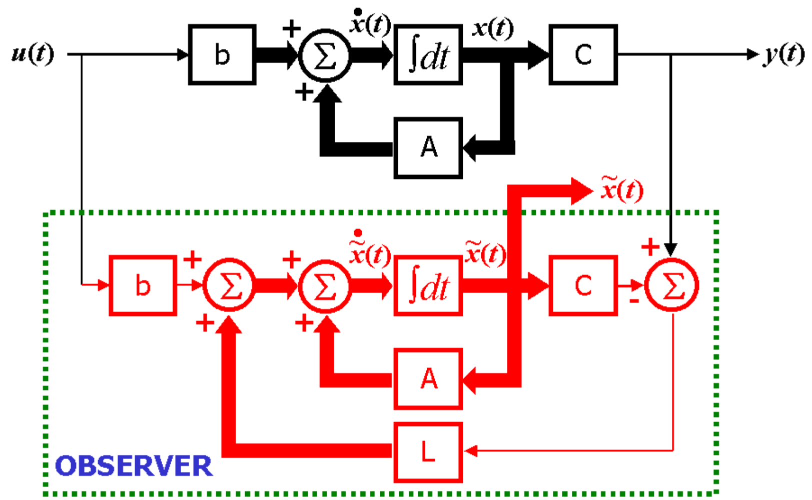
- **First of all, you must notice that the total number of states in the overall system has increased.**



Observer Design and Observer Based Compensators

- **Why should I use an observer? If I know the system matrices, can't I know the state?**

  No! You have input (u) and output (y), NOT x(0). x(0) may be unknown. You are asked to find out x(t) by starting $\tilde{x}(0)$ from another value, e.g. from $\tilde{x}(0)=0$.

Observer Design and Observer Based Compensators

- **State observers can be designed if and only if the observability condition is satisfied.**

Calculate W_o →

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If rank(W_o)=n then the system is said to be completely observable.

Observer Design and Observer Based Compensators

$$\dot{x} = Ax + bu$$
$$y = Cx$$

System

Correction term

$$\dot{\tilde{x}} = A\tilde{x} + bu + L(y - C\,\tilde{x})$$

Observer

Define the error $e = x - \tilde{x}$

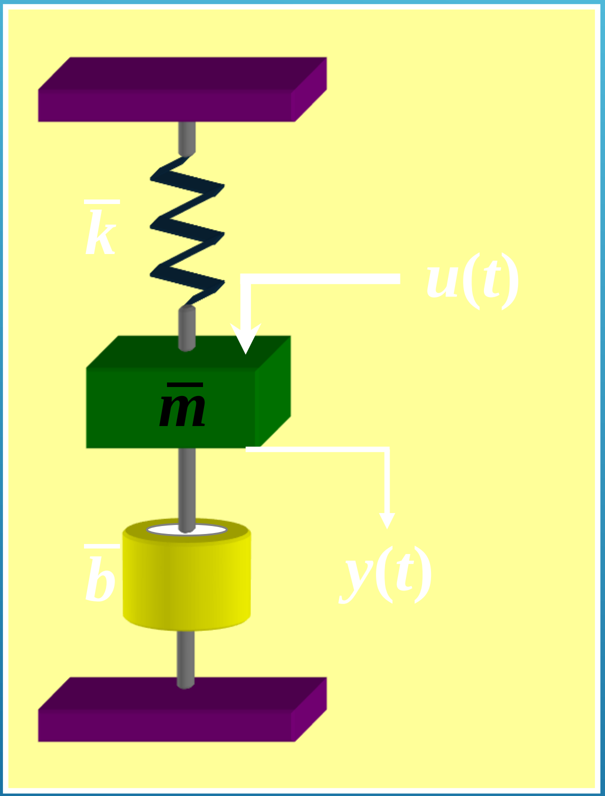
$$\dot{e} = \left(Ax + bu\right) - \left(A\tilde{x} + bu + L(y - C\,\tilde{x})\right)$$

$$\dot{e} = A\left(x - \tilde{x}\right) - L\left(Cx - C\,\tilde{x}\right)$$

$$\dot{e} = \left(A - LC\right)e$$

If the matrix A-LC is stable, then no matter what the initial conditions are, i.e. x(0) and x̃(0); any error vector (say e(0)) will tend to zero, and the observer will generate the state x(t) ultimately.

An Example
Remember, we have studied this before...



Dynamics

$$m\ddot{y} + b\dot{y} + ky = u$$

State

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = Ax + bu$$
$y = Cx$ [redacted]

Let's choose, $\bar{b}$=2, $\bar{m}$=1 and $\bar{k}$=2 (in MKS units...)

Observer

Error eqn.

Matrix to analyze

Char. polynomial

Routh test to fix regions of l_1 and l_2

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u+\begin{bmatrix}l_1\\ l_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}x_1\\ x_2\end{bmatrix}-\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}\right)$$

$$\begin{bmatrix}\dot{x}_1-\dot{\tilde{x}}_1\\ \dot{x}_2-\dot{\tilde{x}}_2\end{bmatrix}=\left(\begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}-\begin{bmatrix}l_1\\ l_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\right)\begin{bmatrix}x_1-\tilde{x}_1\\ x_2-\tilde{x}_2\end{bmatrix}$$

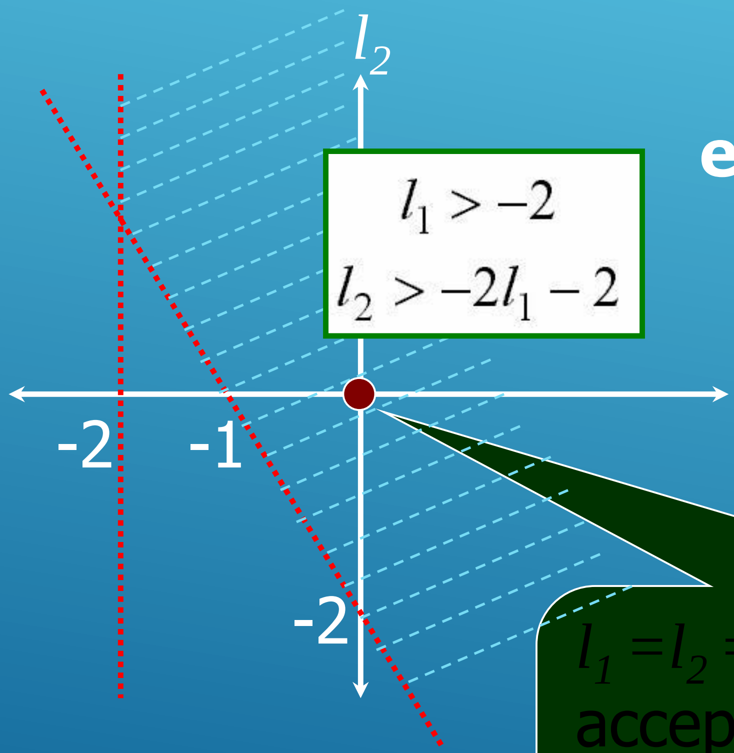
$$A-LC=\begin{bmatrix}-l_1 & 1\\ -2-l_2 & -2\end{bmatrix}$$

$$\left|sI-(A-LC)\right|=\begin{vmatrix}s+l_1 & -1\\ 2+l_2 & s+2\end{vmatrix}=s^2+(2+l_1)s+(2l_1+l_2+2)$$

$$\left.\begin{matrix}s^2 & 1 & (2l_1+l_2+2)\\ s & (2+l_1) & 0\\ 1 & (2l_1+l_2+2) & \end{matrix}\right\}\begin{matrix}2+l_1>0\\ 2l_1+l_2+2>0\end{matrix}\Bigg\}\begin{matrix}l_1>-2\\ l_2>-2l_1-2\end{matrix}$$

An Example

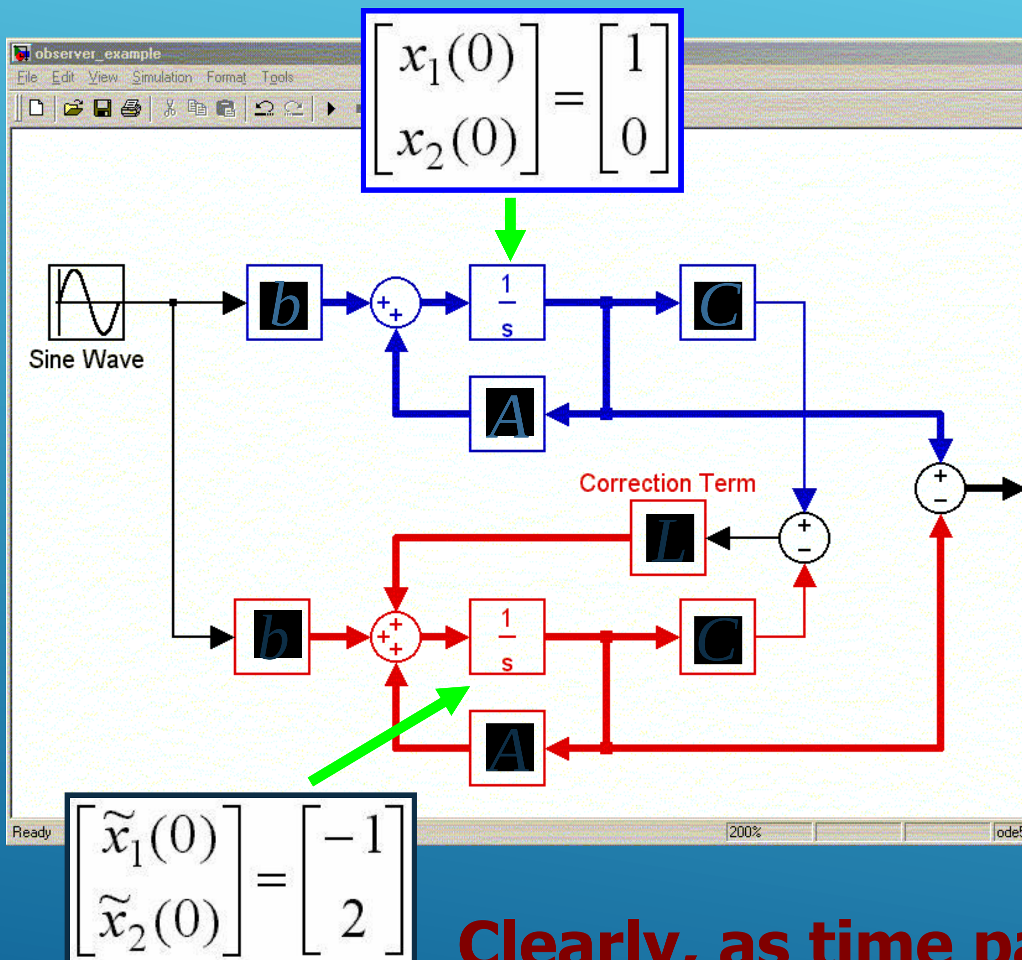
Let's choose $l_1=1$, $l_2=2$
and see what happens…

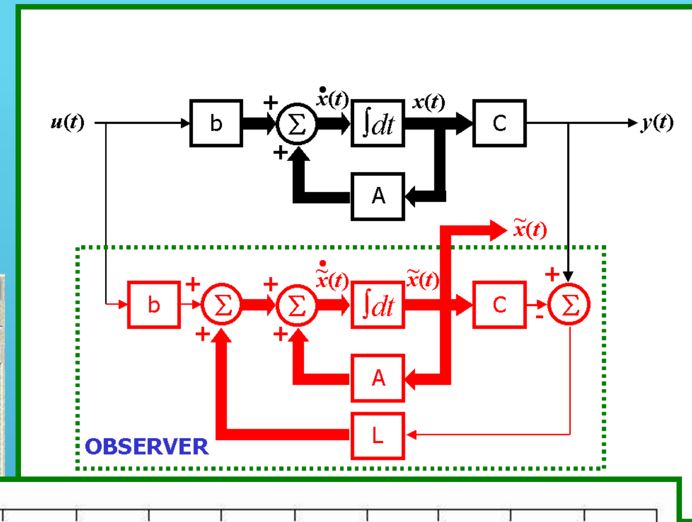


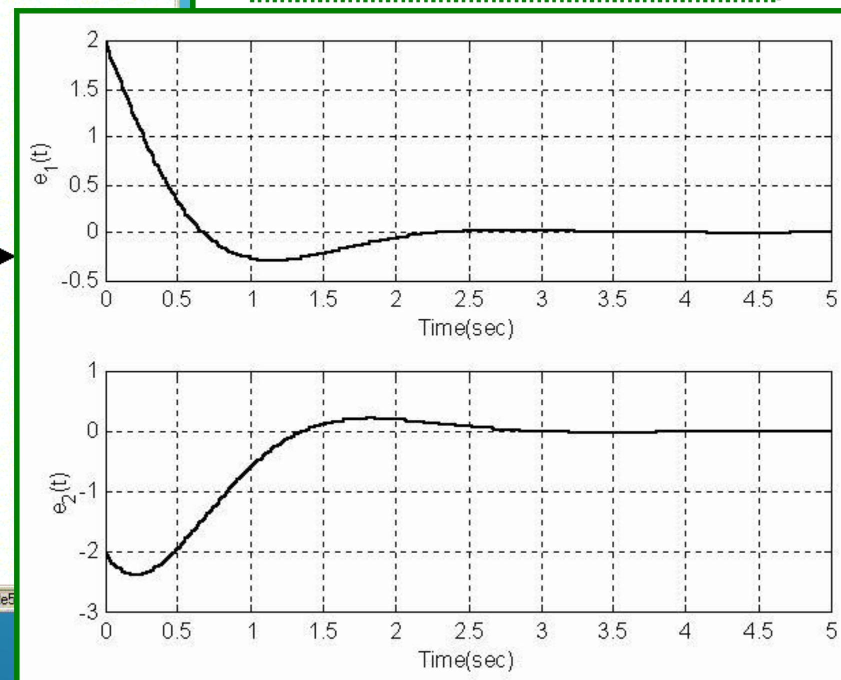
$$\text{eig}(A-LC)=\left\{\begin{array}{l}-1.5000 + j1.9365\\ -1.5000 - j1.9365\end{array}\right\}$$

$l_1=l_2=0$ (i.e. the origin) seems acceptable but, in this case you have no corrective action! Origin seems fine since A is stable!

An Example



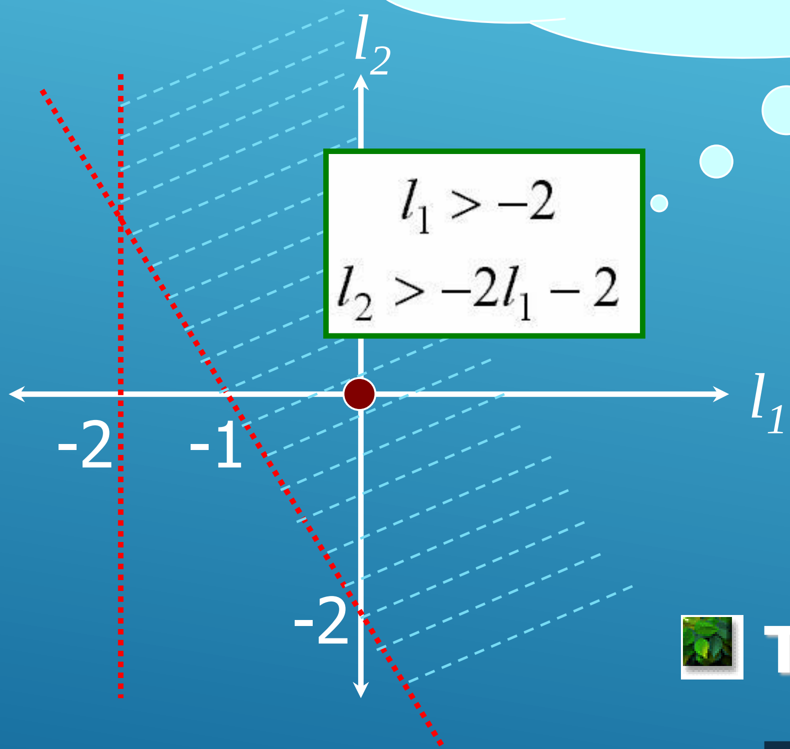




Clearly, as time passes, the state vector of the observer converges to the mass-spring-damper system's state vector…

An Example

Would it be so straightforward
if we had more state variables?



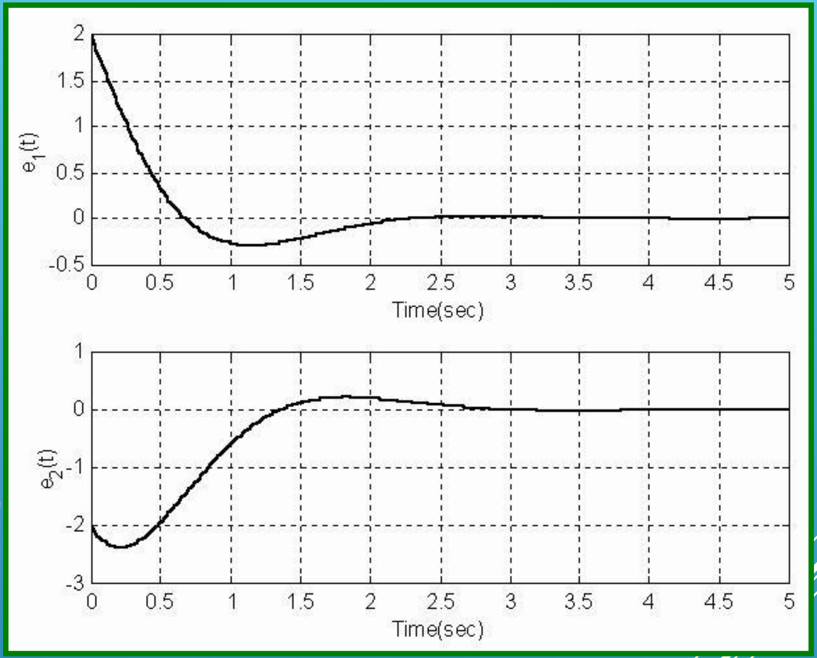
$$l_1 > -2$$
$$l_2 > -2l_1 - 2$$

- **The answer is obviously no!**

Then how to choose L?

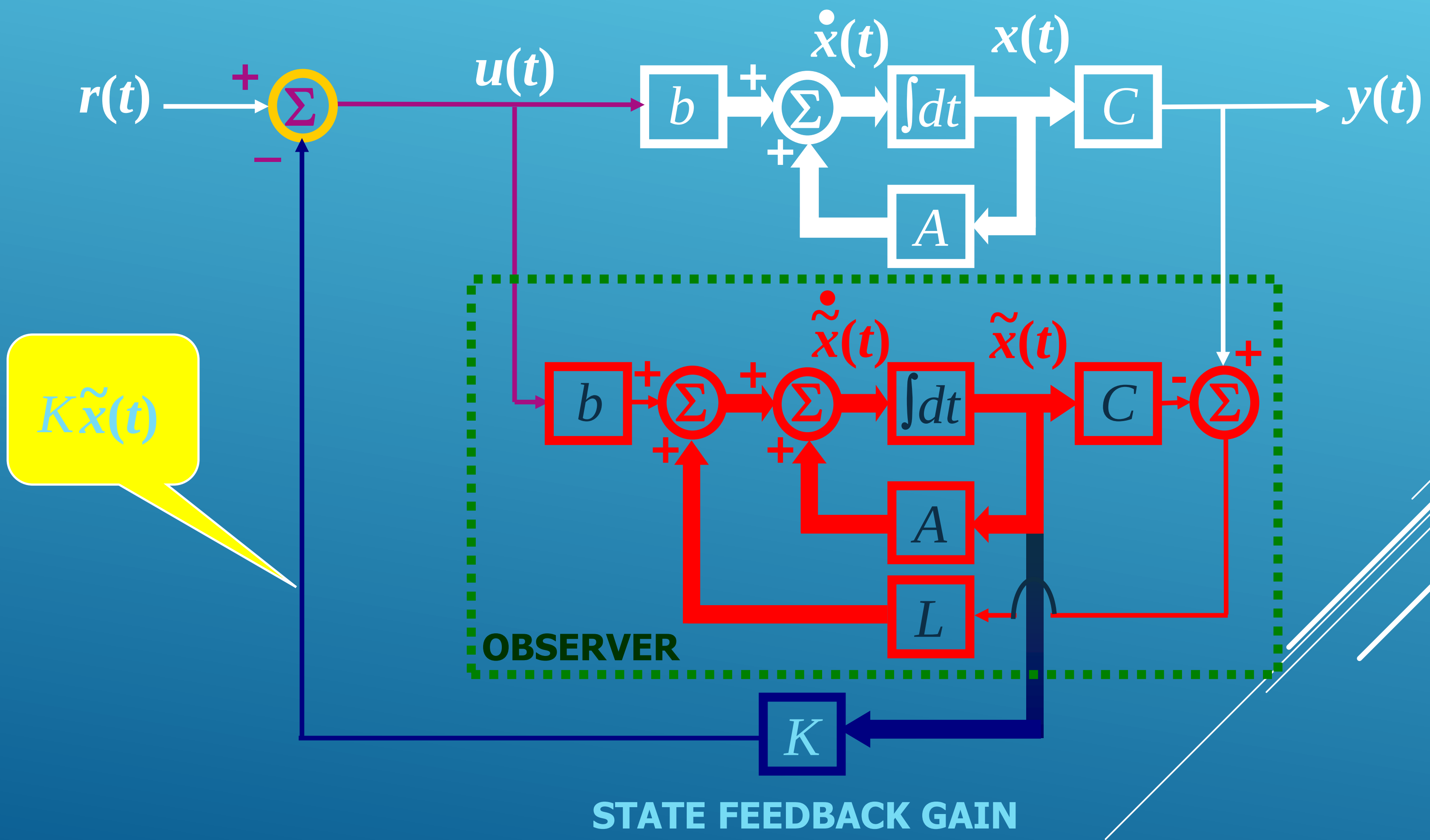
An Example - Remarks

- How satisfactory is this?
- What sorts of design specifications can we have on the response of an observer?
- What would be our strategy to meet those specs?



As a matter of fact, we do not choose L arbitrarily, we design it according to what we need!

Observer Design and Observer Based Compensators



Observer Design and Observer Based Compensators

System

$$\dot{x} = Ax + bu$$
$$y = Cx$$

Observer

$$\dot{\tilde{x}} = A\tilde{x} + bu + L\left(y - C\tilde{x}\right)$$

State Feedback Control Law

$$u = -K\tilde{x} + r$$

$$\dot{x} = Ax - bK\tilde{x} + br$$
$$\dot{\tilde{x}} = (A - bK - LC)\tilde{x} + LCx + br$$

Closed loop dynamics

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A & -bK \\ LC & A - bK - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} b \\ b \end{bmatrix} r$$

A_{CL}

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

Closed loop dynamics (in matrix form)

Observer Design and Observer Based Compensators

$$|sI - A_{CL}| = \begin{vmatrix} sI - A & bK \\ -LC & sI - (A - bK - LC) \end{vmatrix}$$

Add 1st column to the 2nd column, and write as 2nd column

$$= \begin{vmatrix} sI - A & sI - A + bK \\ -LC & sI - A + bK \end{vmatrix}$$

Subtract 2nd row from the 1st row, and write as first row

$$= \begin{vmatrix} sI - A + LC & 0 \\ -LC & sI - A + bK \end{vmatrix}$$

It is now clear to write the determinant as the product of two terms

$$= |sI - A + bK||sI - A + LC|$$

Observer Design and Observer Based Compensators

$$|sI - A_{CL}| = |sI - A + bK||sI - A + LC|$$

- Thus, eig(A_{CL})={Controller Poles} U {Observer Poles}
- Thus, if the eigenvalues of A-bK and A-LC are stable, then the **internal stability** of the closed loop system is guaranteed.
- The result above shows that the design of the state feedback controller and the design of the observer are separated from each other. This is known as (deterministic) **separation principle**.

Observer Design and Observer Based Compensators

* denotes the conjugate transpose

Given the system

$$\dot{x} = Ax + bu$$
$$y = Cx$$

Write the dual the system

$$\dot{z} = A^* z + C^* \gamma$$
$$h = b^* z$$

Notice the state feedback control law is

$$\gamma = K_{new} z$$

Here is the relation between observer gain and the state feedback controller gain

$$L = K^*_{new}$$

Find K^*_{new} by using either Bass-Gura or Ackermann formulas...

[redacted]

Observer Design and Observer Based Compensators

Using the duality property

$$\dot{x} = Ax + bu$$
$$y = Cx$$

$$\dot{z} = A^* z + C^* \gamma$$
$$h = b^* z$$

$$\gamma = K_{new} z$$

$$L = K_{new}^*$$

Bass-Gura

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$
$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$
$$K = [\alpha_1 - a_1 \quad \cdots \quad \alpha_n - a_n]\Omega^{-1} W_c^{-1}$$
$$\Omega = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ 0 & 0 & 0 & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

[redacted] **Ackermann**

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$
$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$
$$K = [0 \quad 0 \quad \cdots \quad 0 \quad 1]_{1\times n} W_c^{-1} \alpha(A)$$
$$\alpha(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

An Example

For the system

$$\dot{x} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 1 \\ 4 & -1 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

- Design an observer such that eig(A-LC)={-5,-5,-5}

- This is equivalent to find the state feedback gain for the following system:

$$\dot{z} = A^* z + C^* \gamma$$

$$h = b^* z$$

$$\gamma = K_{new} z$$

$$L = K_{new}^*$$

An Example

$$a(s) = |sI - A| = s^n + a_1 s^{n-1} + \cdots + a_n$$

$$\alpha(s) = |sI - (A - bK)| = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$$

$$K = [\alpha_1 - a_1 \quad \cdots \quad \alpha_n - a_n]\Omega^{-1} W_c^{-1}$$

$$\Omega = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & 1 & \cdots & a_{n-3} \\ 0 & 0 & 0 & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

MATLAB Editor/Debu...

```
clear all
close all
clc

A = [-1 2 0; 0 -2 1; 4 -1 -5];
b = [0 1 1]';
C = [1 1 1];
d = 0;

[numOL,denOL] = ss2tf(A,b,C,d);

h = tf(numOL,denOL)
roots(denOL)

Wc = [b A*b A*A*b];
Wo = [C;C*A;C*A*A];
[rank(Wc) rank(Wo)]

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp(' Bass-Gura Formula')
alpha = [1 15 75 125]; %Desired |A-LC| roots are specified here
a      = denOL;
Omega = [1 a(2) a(3);0 1 a(2);0 0 1];

Knew = (alpha(2:4)-a(2:4))*inv(Omega)*inv([C' A'*C' A'*A'*C']);
L = Knew'
eig(A-L*C)
```

find_L_exam...

Ready 8:31 PM

MATLAB Command Window

```
Transfer function:
  2 s^2 + 11 s + 27
----------------------
s^3 + 8 s^2 + 18 s + 3


ans =

  -3.9096 + 1.1406i
  -3.9096 - 1.1406i
  -0.1809


ans =

     3     3

 Bass-Gura Formula

L =

    3.5692
    1.3385
    2.0923


ans =

  -5.0000
  -5.0000 + 0.0000i
  -5.0000 - 0.0000i
```

Ready

Observer Design and Observer Based Compensators

Transfer function realization

$$\dot{x} = Ax + bu$$

$$y = Cx$$

$$\dot{\tilde{x}} = A\tilde{x} + bu + L\left(y - C\tilde{x}\right)$$

$$u = -K\tilde{x} + r$$

$$\dot{\tilde{x}} = A\tilde{x} + b\left(-K\tilde{x} + r\right) + L\left(y - C\tilde{x}\right) \text{ or}$$

$$\dot{\tilde{x}} = \left(A - bK - LC\right)\tilde{x} + Ly + br \quad \text{Now take the Laplace transform}$$

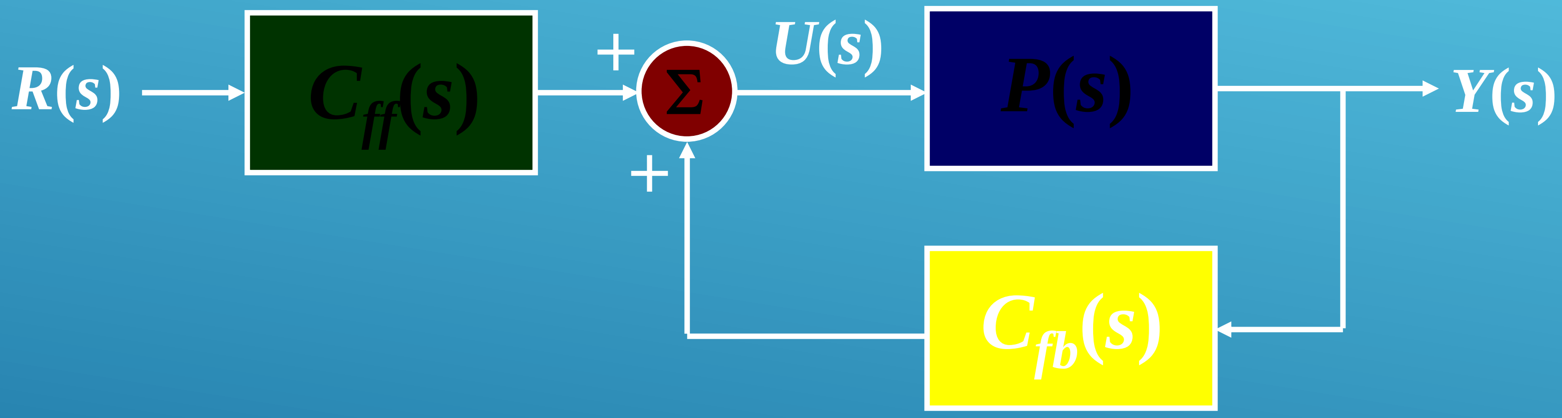
$$\left(sI - A + bK + LC\right)\tilde{X}(s) = LY(s) + bR(s)$$

$$\tilde{X}(s) = \left(sI - A + bK + LC\right)^{-1}\left(LY(s) + bR(s)\right) \text{ Insert this into } U(s)$$

$$\begin{aligned}
U(s) &= -K\tilde{X}(s) + R(s) \\
&= -K\left(sI - A + bK + LC\right)^{-1}\left(LY(s) + bR(s)\right) + R(s) \\
&= -K\left(sI - A + bK + LC\right)^{-1}LY(s) + \left(1 - K\left(sI - A + bK + LC\right)^{-1}b\right)R(s) \\
&= C_{fb}(s)Y(s) + C_{ff}(s)R(s)
\end{aligned}$$

Observer Design and Observer Based Compensators

Transfer function realization



$$C_{fb}(s) = -K\left(sI - A + bK + LC\right)^{-1} L$$

$$C_{ff}(s) = 1 - K\left(sI - A + bK + LC\right)^{-1} b$$

An Example

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

Using Bass-Gura formula we get the following...

$$\alpha(s) = s^3 + 3s^2 + 3s + 1$$

$$K = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$\alpha(s) = s^3 + 6s^2 + 12s + 8$$

$$L = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

eig(sI-A+bK)={-1,-1,-1}
eig(sI-A+LC)={-2,-2,-2}
eig(sI-A+bK+LC)={-3,-1.5±j1.3229}

As a rule of thumb, observer must be at least 2 to 5 times faster than the system response. In this example we did not do this.

An Example

Now, let's calculate feedforward and feedback components of the control system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

$$P(s) = \frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

$$C_{fb}(s) = -K\left(sI - A + bK + LC\right)^{-1} L$$

$$C_{ff}(s) = 1 - K\left(sI - A + bK + LC\right)^{-1} b$$

$$L = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

$$C_{fb}(s) = \frac{-4s^2 - 12s - 8}{s^3 + 6s^2 + 13s + 12}$$

$$C_{ff}(s) = \frac{s^3 + 6s^2 + 12s + 8}{s^3 + 6s^2 + 13s + 12}$$

An Example

Step Input | Ramp Input