Linear process dynamics [1-5]

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Thermometers in liquid

Unsteady state: $$hAT_b - hAT - 0 = mC_p \frac{dT}{dt} \qquad (1)$$

Steady state : $$hAT_{bs} - hAT_s = 0 \qquad (2)$$

Substracting Eq. (2) from Eq. (1);

$$hA(T_b - T_{bs}) - hA(T - T_s) = mC_p \frac{d(T - T_s)}{dt}$$

$$T_b - T_{bs} = T_b' \quad , \quad T - T_s = T'$$

$$hAT_b' - hAT' = mC_p \frac{dT'}{dt}$$

$$hA(T_b' - T') = mC_p \frac{dT'}{dt}$$

$$T_b' - T' = \frac{mC_p}{hA}\frac{dT'}{dt} \qquad \tau = \frac{mC_p}{hA}$$

$$\tau \frac{dT'}{dt} + T' = T_b'$$

Taking the Laplace transform of both sides of the equation, we get;

$$\tau\left[s T^{'}(s) - T^{'}(0)\right] + T^{'}(s) = T_b^{'}(s)$$

$$T^{'}(0) = T(0) - T_S = 0$$

$$\tau\, s T^{'}(s) + T^{'}(s) = T_b^{'}(s)$$

$$\frac{T^{'}(s)}{T_b^{'}(s)} = \frac{1}{\tau\, s + 1}$$

$$T_b^{'}(t) = (80 - 30)\, u(t)$$

$$T_b^{'}(t) = 50\, u(t) \quad \rightarrow \quad T_b^{'}(s) = \frac{50}{s}$$

$$T^{'}(s) = T_b^{'}(s)\frac{1}{\tau\, s + 1}$$

$$T^{'}(s) = \frac{50}{s}\frac{1}{\tau s + 1} = \frac{50/\tau}{s(s + 1/\tau)} = \frac{A}{s} + \frac{B}{s + (1/\tau)}$$

$$A = \left[\frac{\left(50/\tau\right)s}{s(s + (1/\tau))}\right]_{s=0} = 50 \qquad , \qquad B = \left[\frac{(50/\tau)\,(s + (1/\tau))}{s\,(s + (1/\tau))}\right]_{s=-1/\tau} = -50$$

$$T^{'}(s)=\frac{50/\tau}{s(s+(1/\tau))}=\frac{50}{s}-\frac{50}{s+(1/\tau)}$$

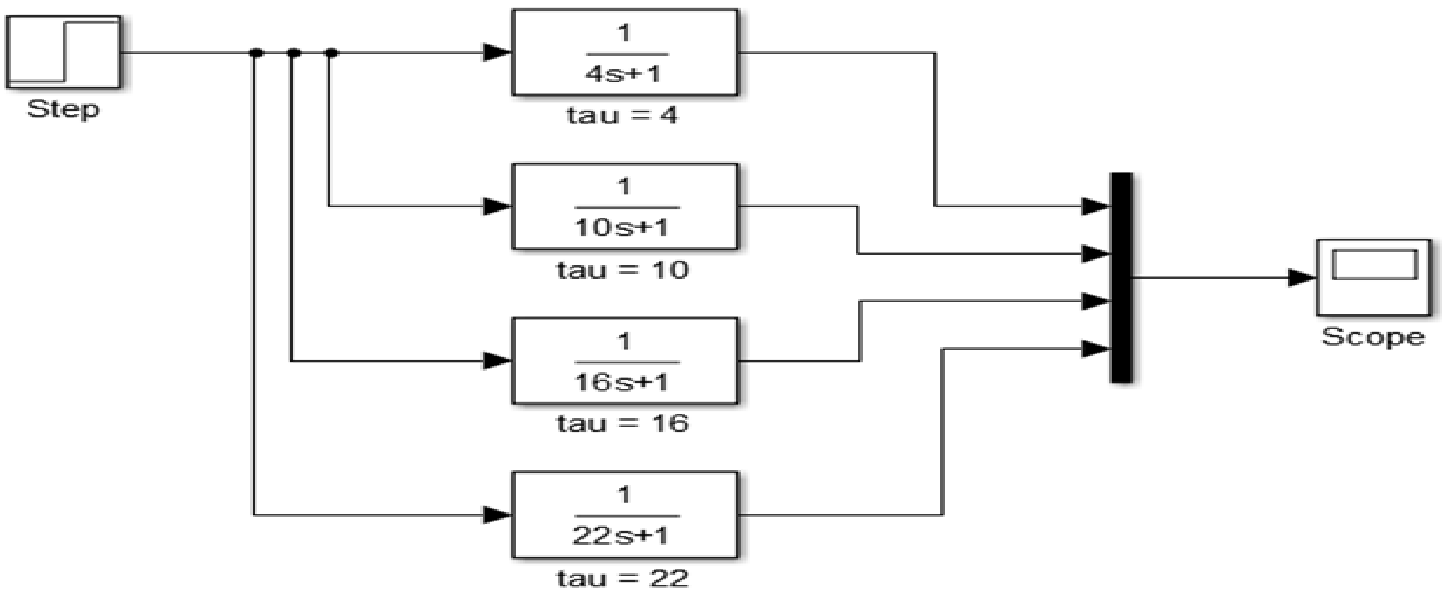
$$T^{'}(t)=50(1-e^{-t/\tau})u(t)$$

$$T^{'}(t)=T(t)-T_s=T(t)-30$$

$$T(t)=30+T^{'}(t)=30+50(1-e^{-t/\tau})u(t)$$

$$t\rightarrow\infty \quad T(\infty)=30+50(1-e^{-t/\tau})=80^{o}C$$

Simulink model:

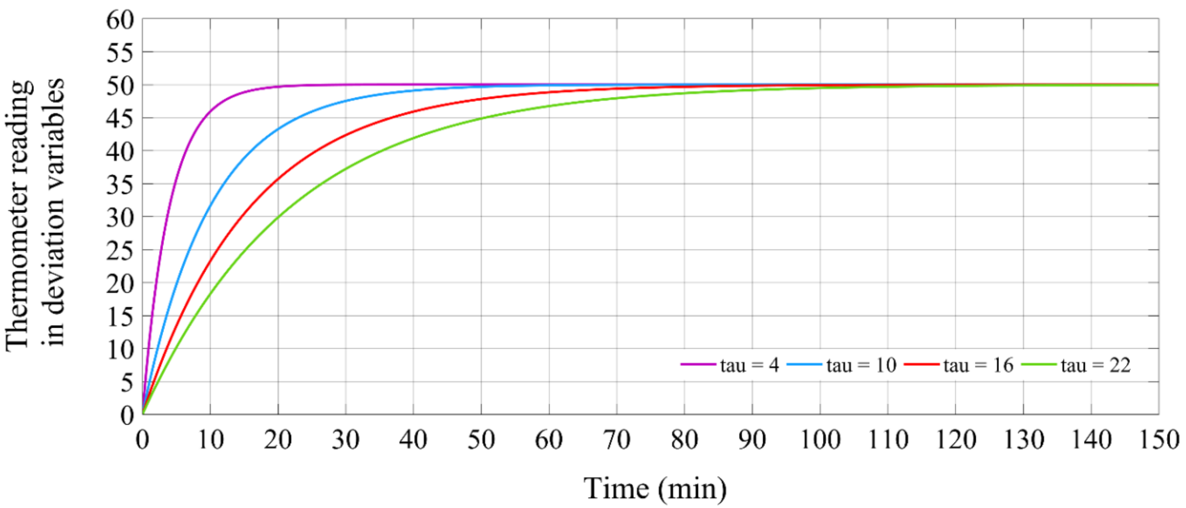


Simulink plot:

As can be seen from the figure; $t \to \infty \quad T^{'}(\infty) = 50(1 - e^{-\infty/\tau})u(t) = 50^{o}C$

$T(t) = 30 + T^{'}(t)$

$t \to \infty \quad T(\infty) = 30 + 50 = 80^{o}C$



According to the results when τ increases, response of the system gets slower.

The first order reaction occurs in two noninteracting reactors connected in series

Assumptions:

F	: Volumetric flow rate
$\rho_{mixture}$	: constant
$T_{mixture}$	: constant
$V_{reactor}$	: constant (V_1=V_2)
m	[=] mol A/min
C_{Ao}	[=] mol A/m^3
C_{A2}	[=] mol A/m^3
(m/ρ_A << F)	

•**Reactor 1:**

Unsteady state

$$V\frac{dC_{A1}}{dt} = F\,C_{Ao} - (F + \frac{m}{\rho_A})C_{A1} - k_1 C_{A1} V + m$$

$$F + \frac{m}{\rho_A} \cong F$$

$$V\frac{dC_{A1}}{dt} + \left(F + k_1 V\right)C_{A1} = F\,C_{Ao} + m$$

Steady state $$0 = F\,C^{o}_{Ao} - \left(F + \frac{m^{o}}{\rho_A}\right)C^{o}_{A1} - k_1 C^{o}_{A1} V + m^{o}$$

$$\left(F + k_1 V\right)C^{o}_{A1} = F\,C^{o}_{Ao} + m^{o}$$

Unsteady-Steady State $$V\frac{dC_{A1}}{dt} + \left(F + k_1 V\right)C_{A1} - \left(F + k_1 V\right)C^{o}_{A1} = FC_{Ao} - FC^{o}_{Ao} + m - m^{o}$$

$$V\frac{d\,(C_{A1} - C^{o}_{A1})}{dt} + \left(F + k_1 V\right)\left(C_{A1} - C^{o}_{1}\right) = F\left(C_{Ao} - C^{o}_{Ao}\right) + \left(m - m^{o}\right)$$

$$C_{A1} - C^{o}_{A1} = C_{A1} \quad ; \quad C_{Ao} - C^{o}_{Ao} = C_{Ao} \quad ; \quad m - m^{o} = M$$

$$V\frac{dC_{A1}}{dt} + \left(F + k_1 V\right)C_{A1} = FC_{Ao} + M$$

Dividing the equation by $(F+k_1V)$;

$$\frac{V}{F + k_1 V}\frac{dC_{A1}}{dt} + C_{A1} = \frac{F}{F + k_1 V}C_{Ao} + \frac{1}{F + k_1 V}M$$

$$\tau_1\frac{dC_{A1}}{dt} + C_{A1} = \frac{1}{1 + k_1\frac{V}{F}}\,C_{Ao} + \frac{1}{F\left(1 + k_1\frac{V}{F}\right)}M$$

Taking the Laplace transform;

$$\tau_1\left[sC_{A1}(s)-C_{A1}(0)\right]+C_{A1}(s)=\frac{1}{1+k_1\frac{V}{F}}C_{Ao}(s)+\frac{1}{F\left(1+k_1\frac{V}{F}\right)}M(s)$$

$$[\tau_1 s+1]C_{A1}(s)=\frac{1}{1+k_1\frac{V}{F}}C_{Ao}(s)+\frac{1}{F\left(1+k_1\frac{V}{F}\right)}M(s)$$

$$C_{A1}(s)=\frac{1/\left(1+k_1\frac{V}{F}\right)}{\tau_1 s+1}C_{Ao}(s)+\frac{1/\left(F\left(1+k_1\frac{V}{F}\right)\right)}{\tau_1 s+1}M(s)$$

Reactor 2:

Unsteady state $V\frac{dC_{A2}}{dt}=F\,C_{A1}-FC_{A2}-k_2C_{A2}V$

$$V\frac{dC_{A2}}{dt}=F\,(C_{A1}-C_{A2})-k_2C_{A2}V$$

Steady state $0=F\,(C_{A1}{}^{o}-C_{A2}{}^{o})-k_2C_{A2}{}^{o}V$

Unsteady – Steady state

$$V\frac{d\,(C_{A2}-C_{A2}^{o})}{dt}=F\,[\left(C_{A1}-C_{A1}^{o}\right)-(C_{A2}-C_{A2}{}^{o})]-k_2V(C_{A2}-C_{A2}{}^{o})$$

$$C_{A1}-C_{A1}^{o}=C_{A1}\qquad;\qquad C_{A2}-C_{A2}^{o}=C_{A2}$$

$$V\frac{dC_{A2}}{dt} = F\,(C_{A1} - C_{A2}) - k_2 C_{A2} V$$

$$V\frac{dC_{A2}}{dt} = F\,C_{A1} - FC_{A2} - k_2 C_{A2} V$$

$$V\frac{dC_{A2}}{dt} + \left(F + k_2 V\right) C_{A2} = FC_{A1}$$

Dividing the equation by ($F+k_2V$);

$$\frac{V}{F + k_2 V}\frac{dC_{A2}}{dt} + C_{A2} = \frac{F}{F + k_2 V} C_{A1}$$

$$\tau_2 \frac{dC_{A2}}{dt} + C_{A2} = \frac{1}{1 + k_2 \frac{V}{F}}\, C_{A1}$$

Taking the Laplace transform;

$$\tau_2\left[sC_{A2}(s) - C_{A2}(0)\right] + C_{A2}(s) = \frac{1}{1 + k_2 \frac{V}{F}}\, C_{A1}(s)$$

$$\left[\tau_2 s + 1\right] C_{A2}(s) = \frac{1}{1 + k_2 \frac{V}{F}} C_{A1}(s)$$

$$C_{A2}(s) = \frac{1/\left(1 + k_2 \frac{V}{F}\right)}{\tau_2 s + 1} C_{A1}(s)$$

Substituting $C_{A1}(s)$ we obtain;

$$C_{A2}(s)=\frac{1/\left(1+k_2\frac{V}{F}\right)}{\tau_2 s+1}\left(\frac{1/\left(1+k_1\frac{V}{F}\right)}{\tau_1 s+1}C_{Ao}(s)+\frac{1/\left(F\left(1+k_1\frac{V}{F}\right)\right)}{\tau_1 s+1}M(s)\right)$$

Noninteracting systems (Systems with no internal interacting)

Here, the second system cannot affect the first system while the first one affects the second system.

The two interacting tanks are connected in series. The cross-sectional areas and the resistances of the tanks are A_1=A_2=2 and R_1=R_2=1

$$R_1 = R_2 = R = 1$$

$$A_1 = A_2 = A = 2$$

Tank 1: $q - q_1 = A_1 \frac{dh_1}{dt}$ $\qquad q_1 = \frac{1}{R_1}(h_1 - h_2)$

Tank 2: $q_1 - q_2 = A_2 \frac{dh_2}{dt}$ $\qquad q_2 = \frac{h_2}{R_2}$

Steady state: $q_s - q_{1s} = 0$

$$q_{1s} - q_{2s} = 0$$

Tank 1: $Q - Q_1 = A_1 \frac{dH_1}{dt}$ $\qquad$ Valve 1: $Q_1 = \frac{H_1 - H_2}{R_1}$

Tank 2: $Q_1 - Q_2 = A_2 \frac{dH_2}{dt}$ $\qquad$ Valve 2: $Q_2 = \frac{H_2}{R_2}$

Tank 1: $Q(s) - Q_1(s) = A_1 s\, H_1(s)$

Tank 2: $Q_1(s) - Q_2(s) = A_2 s\, H_2(s)$

Valve 1: $R_1\, Q_1(s) = H_1(s) - H_2(s)$

Valve 2: $R_2\, Q_2(s) = H_2(s)$

Tank 1: $Q(s) - \frac{H_1(s)}{R_1} + \frac{H_2(s)}{R_1} = A_1 s\, H_1(s)$

Tank 2: $\frac{H_1(s)}{R_1} - \frac{H_2(s)}{R_1} - \frac{H_2(s)}{R_2} = A_2 s\, H_2(s)$

From Tank 1: $R_1 Q(s) - H_1(s) + H_2(s) = R_1 A_1 s\, H_1(s)$

$$R_1 Q(s) + H_2(s) = (R_1 A_1 s + 1) H_1(s)$$

$$H_1(s) = \frac{R_1 Q(s) + H_2(s)}{R_1 A_1 s + 1}$$

From Tank 2: $\dfrac{H_1(s)}{R_1} = \dfrac{H_2(s)}{R_1} + \dfrac{H_2(s)}{R_2} + A_2 s\, H_2(s)$

$$\frac{H_1(s)}{R_1} = H_2(s)\left[\frac{1}{R_1} + \frac{1}{R_2} + A_2 s\right]$$

$$\frac{Q(s) + (\frac{H_2(s)}{R_1})}{R_1 A_1 s + 1} = H_2(s)\left[\frac{1}{R_1} + \frac{1}{R_2} + A_2 s\right]$$

$\tau_1 = R_1 A_1$ $\qquad\qquad$ $\tau_2 = R_2 A_2$

$$\frac{Q(s)}{\tau_1 s + 1} = H_2(s)\left[\frac{-1}{R_1 \tau_1 s + R_1} + A_2 s + \frac{1}{R_1} + \frac{1}{R_2}\right]$$

$$\frac{Q(s)}{H_2(s)} = H_2(s)\left[-\frac{1}{R_1} + A_2 \tau_1 s^2 + A_2 s + \frac{\tau_1 s + 1}{R_1} + \frac{\tau_1 s + 1}{R_2}\right]$$

$$G(s) = \frac{H_2(s)}{Q(s)} = \frac{R_1 R_2}{-R_2 + R_1 R_2 A_2 \tau_1 s^2 + R_1 R_2 A_2 s + R_2 \tau_1 s + R_2 + R_1 \tau_1 s + R_1}$$

$$G(s)=\frac{H_2(s)}{Q\,(s)}=\frac{R_2}{\tau_2\tau_1 s^2+\tau_2 s+\tau_1 s+R_2A_1 s+1}$$

$\tau_1=R_1A_1=1*2=2$ $\qquad\qquad$ $\tau_2=R_2A_2=1*2=2$

$$G(s)=\frac{H_2(s)/R_2}{Q\,(s)}=\frac{Q_2(s)}{Q\,(s)}=\frac{1}{2*2*s^2+2s+2s+1*2*s+1}=\frac{1}{4s^2+6s+1}$$

$\Delta=b^2-4ac=6^2-4*4*1=20$

$$s_{1,2}=\frac{-b\mp\sqrt{\Delta}}{2a}=\frac{-6\mp\sqrt{20}}{2*4}\qquad s_1=-1.30\qquad s_2=-0.19$$

$$\frac{Q_2(s)}{Q\,(s)}=\frac{1}{4s^2+6s+1}=\frac{0.25}{s^2+1.5s+0.25}$$

$$Q_2(s)=\frac{1}{s}\frac{1}{4s^2+6s+1}=\frac{1}{s}\frac{0.25}{s^2+1.5s+0.25}$$

$$\frac{0.25}{s(s^2+1.5s+0.25)}=\frac{A}{s}+\frac{B}{(s+1.30)}+\frac{C}{(s+0.19)}$$

To obtain A, multiply equation by s and set s = 0

$$s\frac{0.25}{s(s+1.30)(s+0.19)}=s\frac{A}{s}+s\frac{B}{(s+1.30)}+s\frac{C}{(s+0.19)}$$

$A=1$

To obtain B, multiply equation by (s + 1.30) and set s = -1.30

$$(s+1.30)\frac{0.25}{(s+1.30)(s+0.19)}=(s+1.30)\frac{A}{(s+1.30)}+(s+1.30)\frac{B}{(s+0.19)}$$

$B=-0.225$

To obtain C, multiply equation by (s + 0.19) and set s = -0.19

$$(s+0.19)\frac{0.25}{(s+1.30)(s+0.19)}=(s+0.19)\frac{A}{(s+1.30)}+(s+0.19)\frac{B}{(s+0.19)}$$

$C=0.225$

$$Q_2(s)=\frac{1}{s}-\frac{0.225}{(s+1.30)}+\frac{0.225}{(s+0.19)}$$

$$Q_2(t)=1-0.225e^{-1.30\,t}+0.225e^{-0.19\,t}$$