Lecture 9: Linear Transformations and Matrices

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Definition (Matrix representation of a linear transformation)

Let $L : V \to W$ be a linear transformation and consider the ordered basis $S = \{v_1, v_2, ..., v_n\}$ and $T = \{w_1, w_2, ..., w_m\}$ for the vector spaces V and W, respectively. The matrix representation of the linear transformation L with respect to the basis S and T is defined by

$$A = \left[[L(v_1)]_T \, [L(v_2)]_T \ldots [L(v_n)]_T \right]_{m \times n} .$$

Also for $v \in V$, we have

$$[L(v)]_T = A \, [v]_S .$$

Linear Transformations and Matrices

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}$ be a linear transformation and consider the standard basis

$$S = \left\{ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$T = \left\{ w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

for the vector spaces $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.
Find the matrix representation of the linear transformation L with respect to the basis S and T.

Solution:

$$\begin{aligned} L(v_1) &= a_1 \odot w_1 \oplus a_2 \odot w_2 \Rightarrow [L(v_1)]_T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ L(v_2) &= b_1 \odot w_1 \oplus b_2 \odot w_2 \Rightarrow [L(v_2)]_T = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ L(v_3) &= c_1 \odot w_1 \oplus c_2 \odot w_2 \Rightarrow [L(v_3)]_T = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \end{aligned}$$

$$A = [[L(v_1)]_T \, [L(v_2)]_T \, [L(v_3)]_T]_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}.$$

Theorem

Let $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation and consider the standard basis $\{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$. Let $A = [L(e_1)\, L(e_2) \ldots L(e_n)]_{m \times n}$. The matrix A is the only matrix satisfiying the property;

$$L(x) = Ax, \text{ for } x \in \mathbb{R}^n.$$

It is called the standard matrix representation of the linear transformation L.

Note that there is a one-to-one correspondence between the linear transformation L and the matrix A, that is;

- if A is $m \times n$ matrix, then there is a corresponding linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ which is defined by $L(x) = Ax$, for $x \in \mathbb{R}^n$.

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- if A is $m \times n$ matrix, then there is a corresponding linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ which is defined by $L(x) = Ax$, for $x \in \mathbb{R}^n$.
- Conversely, if $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is a corresponding $m \times n$ matrix A which is defined by $A = \left[L(e_1)\, L(e_2) \ldots L(e_n) \right]_{m \times n}$.

Example

Find the linear transformation which corresponds to the matrix
$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}_{2\times 3}$.

Solution: The corresponding linear transformation is defined by
$L : \mathbb{R}^3 \to \mathbb{R}^2$,

$$\begin{aligned} L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}. \end{aligned}$$

Linear Transformations and Matrices

- To find the rank of the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, it is enough to check the rank of the matrix A. The rank of an $m \times n$ matrix A is the number of nonzero rows in the reduced row echelon form of the matrix A.

$$\begin{aligned} \dim \mathbb{R}^n &= \dim KerL + \dim L\left(\mathbb{R}^n\right) \\ n &= nullity\ \ A + rank\ \ A \end{aligned}$$

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- If A is an $n \times n$ matrix, then we have

$$rank\ A = n \Leftrightarrow nullity\ A = 0 \Leftrightarrow \det A \neq 0 \Leftrightarrow A^{-1} \text{ exists.}$$

Example

Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix}$.

Solution: If we transform the matrix A to the reduced row echelon form, we have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & -3 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} rankA &= \text{number of nonzero rows of the matrix A in the reduced ref.} \\ &= 2 \end{aligned}$$

$nullityA = n - rankA = 3 - 2 = 1.$

Theorem

Let $L : V \to W$ be a linear transformation and consider the ordered basis $S = \{v_1, v_2, ..., v_n\}$ and $S^{'} = \{v_1', v_2', ..., v_n'\}$ for the vector space V, and $T = \{w_1, w_2, ..., w_m\}$ and $T' = \{w_1', w_2', ..., w_m'\}$ for the vector space W. Let , the transition matrix from basis S' to S is P, and the transition matrix from basis T' to T is Q. If A is the matrix representation for the linear transformation L with respect to the basis S and T, then $Q^{-1}AP$ is the matrix representation for the linear transformation L with respect to the basis S' and T'.

Definition

Let A and B are $n \times n$ matrices, if there exist nonsingular matrix P such that $B = P^{-1}AP$, then it is called B is similar to A.

Theorem

If A and B are similar $n \times n$ matrices, then $rankA = rankB$.