Lecture 11: Diagonalization

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Diagonalization

Definition

The $n \times n$ matrix A is diagonalizable if there exits nonsingular matrix P such that $P^{-1}AP = D$, where $D := \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$ is diagonal matrix.

Diagonalization

Definition

Let $L : V \to V$ be a linear transformation and $dimV = n$. We say that L is diagonalizable, if its matrix representation A is diagonalizable.

Theorem

Let $L : V \to V$ be a linear transformation and $dimV = n$. Then L is diagonalizable $\Leftrightarrow$ V has a basis S which consists of the eigenvectors of L. Moreover, if the matrix representation of L with respect to the basis S is the diagonal matrix D, then the entries on the main diagonal of D are the eigenvalues of L.

Diagonalization

Following theorem gives when an $n \times n$ matrix A can be diagonalized.

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2. *If the roots of the characteristic polynomial of an $n \times n$ matrix A are distinct, then A is diagonalizable.*

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2. *If the roots of the characteristic polynomial of an $n \times n$ matrix A are distinct, then A is diagonalizable.*
3. *If the roots of the characteristic polynomial of an $n \times n$ matrix A are not all distinct, then A may or may not be diagonalizable.*

The procedure for diagonalization

Let A be $n \times n$ matrix.

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4. Construct the matrix P whose columns are eigenvectors of A.
5. Construct the diagonal matrix D such that $P^{-1}AP = D$.

Diagonalization

Example

Diagonalize the matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$, if possible.

Solution:

1. The eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

2. The eigenvectors associated with the eigenvalues are

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 10 \\ 5 \\ 1 \end{bmatrix}.$

3. Since the number of linear independent eigenvectors is equal to the dimension of A, A is diagonalizable.

Diagonalization

4. The matrix P consists of the eigenvectors of A, i.e.

$$P = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. The diagonal matrix D is

$$\begin{aligned} P^{-1}AP &= D \\ &= \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

Applications of diagonalization:

1. $A^{-1} = PD^{-1}P^{-1}; D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1/d_n \end{bmatrix}$

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2. $A^k = PD^kP^{-1}; D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n^k \end{bmatrix}.$

Example

Compute A^5, for the matrix $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: Since A is diagonalizable, we have

$$\begin{aligned} A^5 &= PD^5P^{-1} \\ &= \begin{bmatrix} 1 & 4 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{bmatrix} \begin{bmatrix} 1 & -4 & 10 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 124 & 1800 \\ 0 & 32 & 1055 \\ 0 & 0 & 243 \end{bmatrix}. \end{aligned}$$

Diagonalization

Theorem

If A is real and symmetric matrix, then A is always diagonalizable.
($\exists$ orthogonal matrix P such that $P^T AP = P^{-1}AP = D$.)

Jordan Canonical Form

If an $n \times n$ matrix A cannot be diagonalized, then we can often find a matrix J similar to A. The square matrix J is said to be in Jordan canonical form, and the square matrix J_i is called a Jordan blok.

$$Q^{-1}AQ = J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}, \text{ where } J_i := \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}.$$