Logic

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Definitions

- Its origin : logos (Greek), 'the word' or 'what is spoken'
- 'thought' or 'reason'

- 'art of reason', or 'science of reasoning'
- systematic study of the form of valid arguments

(study of the difference between valid arguments and invalid arguments)

(finding out what it is that makes a valid argument valid)

Definitions

argument : sequence of sentences (propositions); premises at the beginning and conclusion at the end

if the premises are all true, then the conclusion must be true

1) All men are mortal
   Socrates is a man premises

   Socrates is mortal conclusion

2) John will come to the party, or Mary will come to the party
   John will not come to the party

   Mary will come to the party

Definitions

Proposition : a sentence that states a fact, true or false (not both)
(the thruthness of the sentence can be evaluated)

- Istanbul is the biggest city of Turkey
- 2 + 3 = 5
- 2 + 1 = 4
- Antalya is the capital city of Turkey
- 2 + x = 8
- Ankara is the best place to live on Earth

letters p, q, r, s are mostly used to represent propositional variables

most of the mathematical statements are constructed by combining one or more propositions using logical operators
(connectives)

Logical Operators

Negation (~p) : "it's not the case that p" or "not p" .

- p : 2 + 3 = 5,

  ~p : it is not the case that 2 + 3 = 5
  ~p : 2 + 3 ≠ 5

p	~p
T	F
F	T

Logical Operators

Conjunction ($p \land q$) : "p and q" .

- p : Ali passed the course
  q : Hasan passed the course

  $p \land q$: Ali and Hasan both passed the course.

p	q	$p \land q$
T	T	T
T	F	F
F	T	F
F	F	F

Logical Operators

Disjunction (p∨q) : "p or q" .

- p : *Ali* passed the course
  q : Hasan passed the course

  p ∨ q : *Ali* or Hasan passed the course.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Logical Operators

Exclusive or ($p \oplus q$) : "p exclusive or q" .

- p : Ali passed the course
  q : Hasan passed the course

  $p \oplus q$: Ali or Hasan, but not both, passed the course.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical Operators

Conditional Statement ($p \rightarrow q$) : “if p, then q” (p implies q).

- p : it rains
  q : the ground is wet

  p → q : If it rains, then the ground will be wet.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Operators

Biconditional Statement (p↔q) : "p if and only if q" (p implies q and q implies p).

- p : you can take the flight
  q : you have a ticket

p ↔ q : you can take the flight if and only if you have a ticket.

p	q	p → q	q → p	p ↔ q
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

Logical Operators

Truth Tables

$(p \vee \sim q) \rightarrow (p \wedge q)$

p	q	$\sim q$	$p \wedge q$	$p \vee \sim q$	$(p \vee \sim q) \rightarrow (p \wedge q)$
1	1	0	1	1	1
1	0	1	0	1	0
0	1	0	0	0	1
0	0	1	0	1	0

Logical Operators

Truth Tables

$q \leftrightarrow (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$q \leftrightarrow (\sim p \vee \sim q)$
1	1	0	0	0	0
1	0	0	1	1	0
0	1	1	0	1	1
0	0	1	1	1	0

Logical Operators

Truth Tables

$p \rightarrow (p \vee q)$ $p \wedge (\sim p \wedge q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
1	1	1	1	0	0	
1	0	1	1	0	0	
0	1	1	1	1	1	
0	0	0	1	1	0	

Logical Operators

Truth Tables

p → (p∨q)

p ∧ (~p ∧ q)

p	q	p∨q	p → (p∨q)	~p	~p∧q	p ∧ (~p ∧ q)
1	1	1	1	0	0	0
1	0	1	1	0	0	0
0	1	1	1	1	1	0
0	0	0	1	1	0	0

- *A* compound proposition is called tautology if it's true for all the cases

- *A* compound proposition is called contradiction if it's false for all the cases

Logical Equivalences

- If the compound propositions p and q have same truth values for all the cases, they are called logically equivalent

p	q	~p	~pVq	p → q
1	1	0	1	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

$\sim p \vee q \equiv p \rightarrow q$

De Morgan's Low

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	~p	~q	~(p ∧ q)	~pV~q
1	1	0	0	0	0
1	0	0	1	1	1
0	1	1	0	1	1
0	0	0	1	1	1

Logical Equivalences

- De Morgan's Low

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

- $\sim(\sim p) \equiv p$

- $p \wedge 1 \equiv p$

  $p \vee 0 \equiv p$

- $p \wedge 0 \equiv 0$

  $p \vee 1 \equiv 1$

- $p \wedge p \equiv p$

  $p \vee p \equiv p$

- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

  $(p \vee q) \vee r \equiv p \vee (q \vee r)$

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- $p \wedge \sim p \equiv 0$

  $p \vee \sim p \equiv 1$

- $p \rightarrow q \equiv \sim p \vee q$

  $p \rightarrow q \equiv \sim q \rightarrow \sim p$

Logical Equivalences

- $$\begin{aligned} \sim(p \vee (\sim p \wedge q) &\equiv \sim p \wedge \sim(\sim p \wedge q) \\ &\equiv \sim p \wedge (p \vee \sim q) \\ &\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) \\ &\equiv 0 \vee (\sim p \wedge \sim q) \\ &\equiv \sim p \wedge \sim q \end{aligned}$$

- $$\begin{aligned} (p \rightarrow r) \wedge (p \rightarrow q) &\equiv (\sim p \vee r) \wedge (\sim p \vee q) \\ &\equiv \sim p \vee (r \wedge q) \\ &\equiv p \rightarrow (r \wedge q) \end{aligned}$$

Predicates

- p : '2 + 3 = 5'

  q : 'my computer is vulnerable to side channel attacks'

- P(x) : 'x + 3 = 5'

  Q(x) : 'computer x is vulnerable to side channel attacks'

Definition Propositions (or statements) that contains variables

- When a value is assigned to the variable x, then P(x) becomes a proposition and has a truth value.

Predicates

- P(x) : '$x > 3$'
  P(4) is true, but P(2) is false

- Q(x,y) : '$x + 3 = y$'
  Q(4,7) is true, but Q(4,2) is false

- R(x,y,z) : '$x + y = z$'
  R(2,1,3) is true, but R(3,2,2) is false

Quantifiers

- Another way of creating a proposition from a propositional function

Universal Quantifier

Q : ∀x P(x)	If P(x) is true for all x in the domain, then Q is true If there is an x_0 such that P(x_0) is not true, then Q is false

Existential Quantifier

R : ∃x P(x)	If there exists an x_0 such that P(x_0) is true, then R is true If P(x) is false for all x in the domain, then R is false

Quantifiers

- P(x) : $x^2 \geq x$

  What is the truth value of ∀x P(x) if the domain is Z^+ ?

  For all $x \in Z^+$ $x^2 \geq x$. So ∀x P(x) is true for Z^+ .

- Q(x) : x = x + 1

  What is the truth value of ∃x Q(x) if the domain is R?

  There is no real number x such that x = x + 1. So ∃x Q(x) is false for R.

Quantifiers

- P(x) : $x^2 + 1 < 10$, D = { 1, 2, 3}

  What is the truth value of ∀x P(x) if the domain is D?

  If the domain consists of n elements,
  then ∀x P(x) ≡ P(x_1) ∧ P(x_2) ∧ . . . ∧ P(x_n)

  P(1) : $2 < 10$, true
  P(2) : $5 < 10$, true
  P(3) : $10 < 10$, false

Since 1 ∧ 1 ∧ 0 ≡ 0, then ∀x P(x) is false for D.

Quantifiers

- Q(x) : $x^2 < 3$, D = {1, 2, 3}

  What is the truth value of ∃x Q(x) if the domain is D?

  If the domain consists of n elements,
  then ∃x P(x) ≡ P(x_1) ∨ P(x_2) ∨ . . . ∨ P(x_n)

  P(1) : $1 < 3$, true
  P(2) : $4 < 3$, false
  P(3) : $9 < 3$, false

  Since 1 ∨ 0 ∨ 0 ≡ 1, then ∃x P(x) is true for D.

Quantifiers

- Every student in this class has entered the entrance exam

$\forall x\, P(x)$, ' x has taken the entrance exam'

Negation

- It's not the case that every student in this class has entered the entrance exam.

  There is a student in this class who has not taken the entrance exam.

$$
\begin{aligned}
\sim(\forall x\, P(x)) &\equiv \sim(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\
&\equiv \sim P(x_1) \vee \sim P(x_2) \vee \ldots \vee \sim P(x_n) \\
&\equiv \exists x \sim P(x)
\end{aligned}
$$

Quantifiers

- There is a student in this class who has taken the entrance exam.

  $\exists x\, P(x)$, ' x has taken the entrance exam'

Negation

- It's not the case that There is a student in this class who has taken the entrance exam

  None of the students in this class has taken the entrance exam.

$$
\begin{aligned}
\sim(\exists x\, P(x)) &\equiv \sim(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \\
&\equiv \sim P(x_1) \wedge \sim P(x_2) \wedge \ldots \wedge \sim P(x_n) \\
&\equiv \forall x \sim P(x)
\end{aligned}
$$

Quantifiers

- $\sim(\forall x(x^2 > x)) \equiv \exists \mathrm{x} \sim(x^2 > x)$
  $\equiv \exists \mathrm{x}\ x^2 \leq x$

- $\sim(\exists x(x^2 = 7)) \equiv \forall \mathrm{x} \sim(x^2 = 7)$
  $\equiv \forall \mathrm{x}\ x^2 \neq 7$

Quantifiers

- $\underbrace{\forall x\, \forall y\, \overbrace{((x > 0) \land (y < 0) \rightarrow (xy < 0))}^{\text{If x is positive and y is negative, then xy is negative}}}_{\text{For every real numbers x and y, if x is positive and y is negative, then xy is negative}}$ $D = R$

Quantifiers

- For every two integers, if these integers are both positive, then the sum of these integers is also positive
- For two integers x and y, if $x > 0$ and $y > 0$, then $\mathrm{x} + y > 0$

$$\forall x \, \forall y \, ((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

Quantifiers

- There exist integers x and y such that $x + y = 6$

$$\exists x \, \exists y \, (x + y = 6)$$

or

$$\exists y \, \exists x \, (x + y = 6)$$

- $\forall x \, \exists y \, (x + y = 6)$

  For every integer x, there exists an integer y such that $x + y = 6$ (It's true)

- $\exists \mathrm{y} \, \forall x \, (x + y = 6)$

  There exists an integer y so that for all integers x, $x + y = 6$ (It's false)

Proofs

- Valid arguments that establish the truth of mathematical statements

argument : sequence of sentences (propositions); premises at the beginning and conclusion at the end

Proofs

- An argument is called valid if the truthness of all its premises implies that the conlusion is true

- If you have a password, then you can log onto the network.

$p \rightarrow q$

- You have a password

p

- Therefore, you can log onto the network

$$\frac{\begin{array}{c} p \rightarrow q \\ p \end{array}}{q}$$

Proofs

Modus Ponens

$$\begin{array}{l} p \to q \\ p \\ \hline q \end{array}$$

p	q	$p \to q$	$p \land (p \to q)$	$[p \land (p \to q)] \to q$
1	1	0	0	1
1	0	1	1	1
0	1	1	0	1
0	0	1	0	1

Proofs

Modus Ponens

- If $\sqrt{5} > \sqrt{3}$, then $(\sqrt{5})^2 > (\sqrt{3})^2$.
- We know that $\sqrt{5} > \sqrt{3}$
- So, $(\sqrt{5})^2 > (\sqrt{3})^2 \rightarrow 5 > 3$

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline q \end{array}$$

Proofs

- To prove $\forall x\ (P(x) \rightarrow Q(x))$, show that $P(c) \rightarrow Q(c)$ is true for an arbitrary element c of the domain.

- To prove $P(c) \rightarrow Q(c)$, show that $Q(c)$ is true if $P(c)$ is true (p → q is true unless p is true but q is false)

Direct Proof

- To prove p → q is true, first assume p is true, then show that q must also be true.

- Thus, if p is true, then q must also be true, so that the combination of p true and q false never occurs

Proofs

Direct Proof

$$\text{If } \underbrace{n \text{ is odd integer}}_{p}\text{, then } \underbrace{n^2 \text{ is odd integer}}_{q}.$$

$p \rightarrow q$

assume p is true

$$n = 2k + 1, \exists k \in Z$$
$$n^2 = (2k+1)^2$$
$$n^2 = 4k^2 + 2k + 1$$
$$n^2 = 2(2k^2 + k) + 1$$
$$n^2 = 2m + 1, \exists m \in Z$$

q is also true

Proofs

Direct Proof

$$\text{If } \underbrace{m \text{ and } n \text{ are perfect squares}}_{p}, \text{ then } \underbrace{m.n \text{ is also a perfect square}}_{q}.$$

p → q

assume p is true

$m = x^2$ and $n = y^2, \exists x, y \in Z$

$m.n = x^2y^2$

$m.n = (x.y)^2$

$m.n = k^2, \exists k \in Z$

q is also true

Proofs

Proof by Contraposition

If $\underbrace{3n + 2 \text{ is an odd integer}}_{p}$, then $\underbrace{n \text{ is odd integer}}_{q}$

p → q

assume p is true

$$3n + 2 = 2k + 1, \exists k \in Z$$
$$3n = 2k - 1$$
$$n = \frac{2k-1}{3}$$

Proofs

Proof by Contraposition

$p \rightarrow q \equiv \sim q \rightarrow \sim p$

If $3n + 2$ is an odd integer, then n is odd integer

$$\text{If } \underbrace{\text{n is not odd integer}}_{\sim q}\text{, then } \underbrace{3n + 2 \text{ is not odd integer}}_{\sim p}$$

assume ~q is true

$n = 2k, \exists k \in Z$

$3n + 2 = 6k + 8$

$3n + 2 = 2(3k + 4)$

$3n + 2 = 2m, \exists m \in Z$

~p is also true

Proofs

Proof by Contraposition

$p \rightarrow q \equiv \sim q \rightarrow \sim p$

Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

$$\text{If } \underbrace{x < 50 \text{ and } y < 50}_{\sim q}, \text{ then } \underbrace{x + y < 100}_{\sim p}$$

assume ~q is true

$x < 50$ and $y < 50$

$x + y < 100$

~p is also true

Proofs

Proof by Contradiction

- To prove that 'p is true', find a contradiction q such that ~p → q is true.

$$\sim p \rightarrow q$$

$$F \rightarrow F \equiv \boxed{T}$$

- assuming '~p is true' leads us a contradiction

$$q \equiv r \wedge \sim r \equiv 0$$

Proofs

Proof by Contradiction

- Prove that the sum of an irrational number and rational number is irrational.

Assume that the sum of an irrational number x and a rational number y is rational. (~p is true)

$y = \frac{a}{b}$ and $x + y = \frac{c}{d}$, $\exists\ a, b, c, d \in Z$

There is no integers e,f such that $x = \frac{e}{f}$ (the proposition r)

$x + y = \frac{c}{d} \rightarrow x + \frac{a}{b} = \frac{c}{d} \rightarrow x = \frac{c}{d} - \frac{a}{b} \rightarrow x = \frac{e}{f}$, $\exists\ e, f \in Z$ (~r)

~p → (r ∧ ~r) : assuming '~p is true' leads us a contradiction.

Proofs

Proof by Contradiction

- Prove that if $\underbrace{3n + 2 \text{ is an odd integer}}_{p}$, then $\underbrace{n \text{ is odd integer}}_{q}$

Assuming 'p ∧ ~q is not true' leads us a contradiction.

$3n + 2$ is an odd integer and n is even integer. (p ∧ ~q)

$n = 2k, \exists\, k \in Z$. So $3n + 2 = 6k + 2 = 2(3k + 1) = 2m, \exists\, m \in Z$

$3n + 2$ is an even integer. (Contradiction!)