COUNTING II

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Combinations II

- We are selecting four pieces of fruits from a bowl that contains apples, oranges, and pears. How many different selections can we get ?

4A	3A-1P	2A-2P	2A-1P-1O
4O	3A-1O	2A-2O	2O-1A-1P
4P	3O-1P	2O-2P	2P-1A-1O
	3O-1A		
	3P-1O		
	3P-1A		

Combinations II

- 5 people go to a restaurant for the lunch.
- 3 possible choices : cheeseburger, fish sandwich, durum
- How many different menus can be created for 5 of them ?

CCDDF
CDFFF
FFFFF
CCCCD

C		D		F
XX	l	XX	l	X
X	l	X	l	XXX
	l		l	XXXXX
XXXX	l	X	l	

- divider l and X
- 5 X and 2 l, 7 symbols

$$\frac{7!}{2!.5!} = C(7,5)$$

Combinations II

- r combinations of n elements when repetition is allowed :

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

- 5 people and 3 choices for each (5 combinations of 3 objects)

$$\binom{5+3-1}{5}$$

- Suppose there is a box of bills: 5TL, 10TL, 20TL, 50TL, 100TL, 200TL. If you select 5 bills from this box, how many different selections can you make ?

5 combinations of 6 objects : $\binom{6+5-1}{5} = \binom{10}{5}$

Combinations II

- r combinations of n elements when repetition is allowed :

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

- $x_1 + x_2 + x_3 + x_4 = 7$ where $x_i \geq 0$ for $1 \leq i \leq 4$. How many different integer solution sets are there ?

  $S = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 7 \ \ where \ \ x_i \geq 0 \ \ and \ \ x_i \in Z\}$, |S| = ?

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, and $x_4 = 6$ could be one of the solutions. (0, 1, 0, 6)

(0, 1, 0, 6), (1, 2, 1, 3), (4, 0, 3, 0), . . .

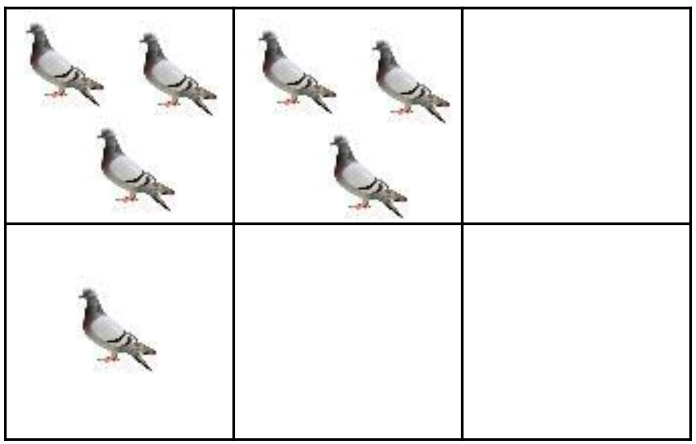
x_1	x_2	x_3	x_4
	x		xxxxxx
x	xx	x	xxx
xxxx		xxx	

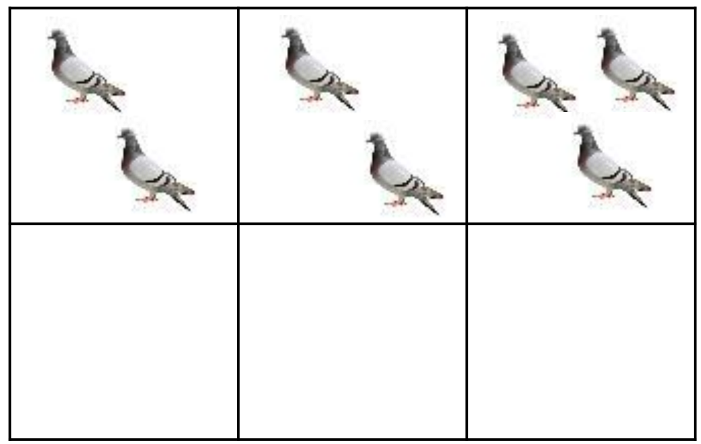
7 combinations of 4 objects

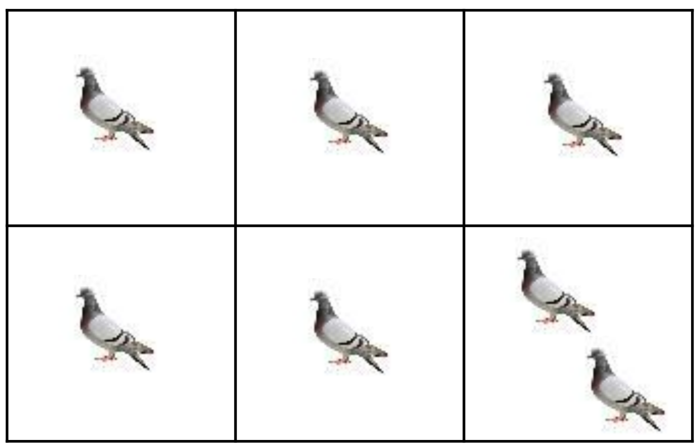
$$\binom{7+4-1}{7} = \binom{10}{7}$$

Pigeonhole Principle

- Assume there are 6 pigeonholes but 7 pigeons, and the pigeons are placed to pigeonholes.







Pigeonhole Principle

- There are 366 days in a year. If there are 367 people, there must be at least two people sharing same birthday.

- If N objects are placed into k boxes, then there is at least one box that contains at least $\lceil N/k \rceil$ objects.

  - $\lceil 37/5 \rceil = 8$

$$\frac{37}{5} = \frac{35}{5} + \frac{2}{5}$$
$$\frac{37}{5} = 7 + 0.4$$

  - $\lceil N/k \rceil = Q + 1$

$$N = Q.k + R$$
$$\frac{N}{k} = Q + \frac{R}{k}$$

Pigeonhole Principle

- Among 50 people, there are at least $\lceil 50/12 \rceil = 5$ people born in the same month

- Assume there are 5 possible grades: A, B, C, D, E. If we want at least 6 students to get same grade on midterm, what should the minimum number of students be ?

$\lceil N/5 \rceil = 6$, then N = 26

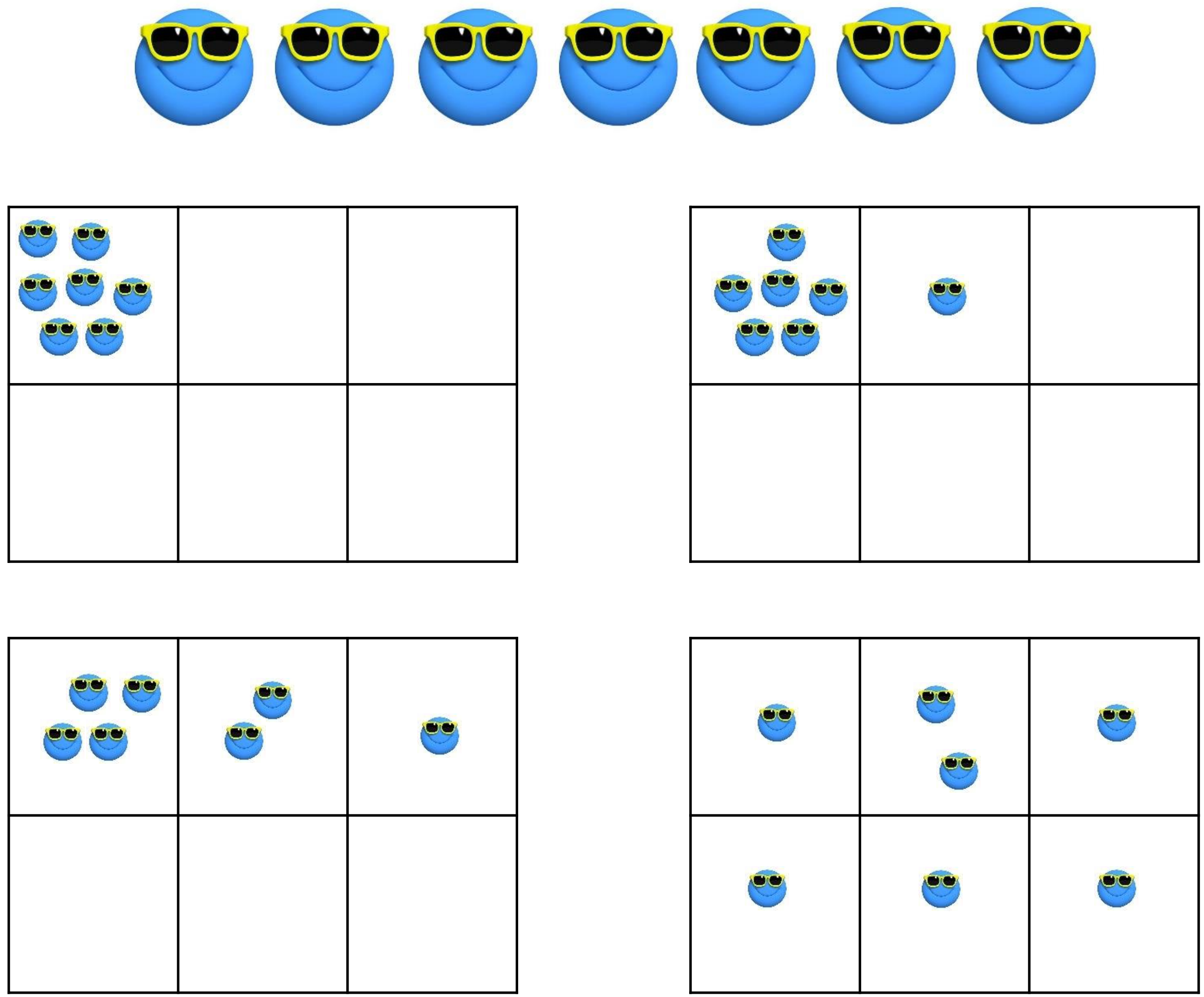
Consider a standard deck of 52 cards:
At least how many cards should be chosen to guarantee that at least 3 cards are chosen from the same suit ?

$\lceil N/4 \rceil = 3$, then N = 9

At least how many cards should be chosen to guarantee that at least 3 clubs are chosen ? (think about the worst case to guarantee that)

all diamonds + all spades + all hearts + 3 clubs = 42

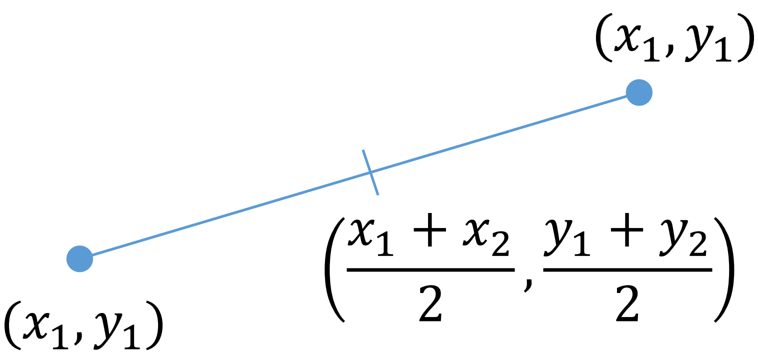
Pigeonhole Principle



Pigeonhole Principle

- Let (x_i, y_i) , $i = 1, 2, 3, 4, 5$ be a set of five distinct points with integer coordinates in the *xy* plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)$$



(E, O)	(O, E)	(E, E)	(O, O)
(x_1, y_1)	(x_2, y_2)	(x_3, y_3) (x_5, y_5)	(x_4, y_4)

- In which condition, $\frac{x_1+x_2}{2} \in Z$
- $\frac{E+E}{2}$ or $\frac{O+O}{2}$
- if there are (E, O) and (E, O), then $\left(\frac{E+E}{2}, \frac{O+O}{2}\right)$ will be integer
- Thus if there are in the same form, mid point will be integer

Pigeonhole Principle

How many ordered pairs of integers (a, b), are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \; mod \; 5 = a_2 \; mod \; 5$ and $b_1 \; mod \; 5 = b_2 \; mod \; 5$?

- $a \;\; mod \;\; 5 = R$ where R is remaninder of the division $(a \, / \, 5)$
- How many remainders are there when an integer is divided by 5 ?

  0, 1, 2, 3, 4

- How many possible pairs of remainders are there ? $(a \; mod \; 5, b \; mod \; 5)$

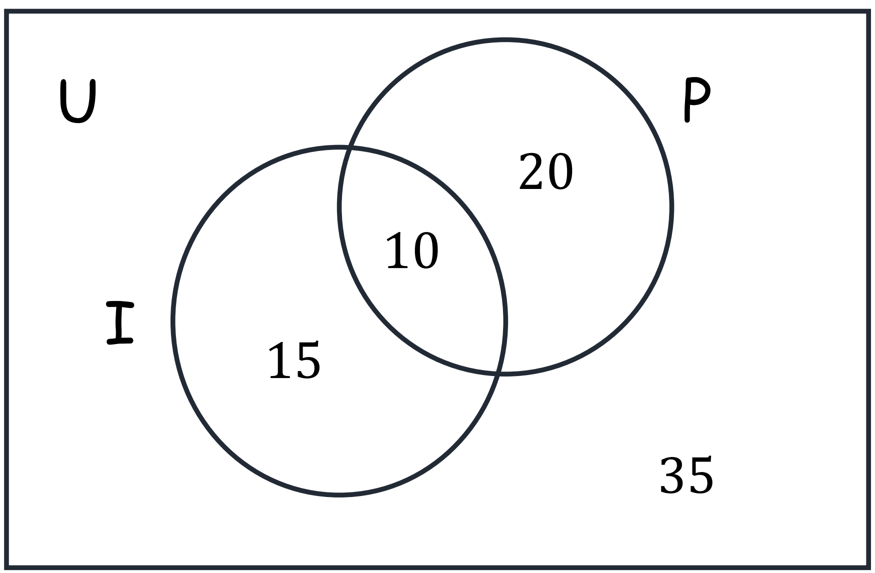
  5 x 5 = 25

Thus, there should be 26 pairs of remainders so that some two pairs (a_1, b_1) and (a_1, b_1) will have same pair of remainders,

$a_1 \; mod \; 5 = a_2 \; mod \; 5$ and $b_1 \; mod \; 5 = b_2 \; mod \; 5$

Principles of Inclusion & Exclusion

Assume there 80 students enrolled to freshman engineering program this year. 30 of them also enrolled in Physics, 25 of them also enrolled in Introdution to Programming, and 10 of them also enrolled in both Physics and Intoduction to Programming courses. How many of them neither taking Physics nor taking Introdution to Programming ?



$|U| = 80$

$|I \cup P| = |I| + |P| - |I \cap P|$

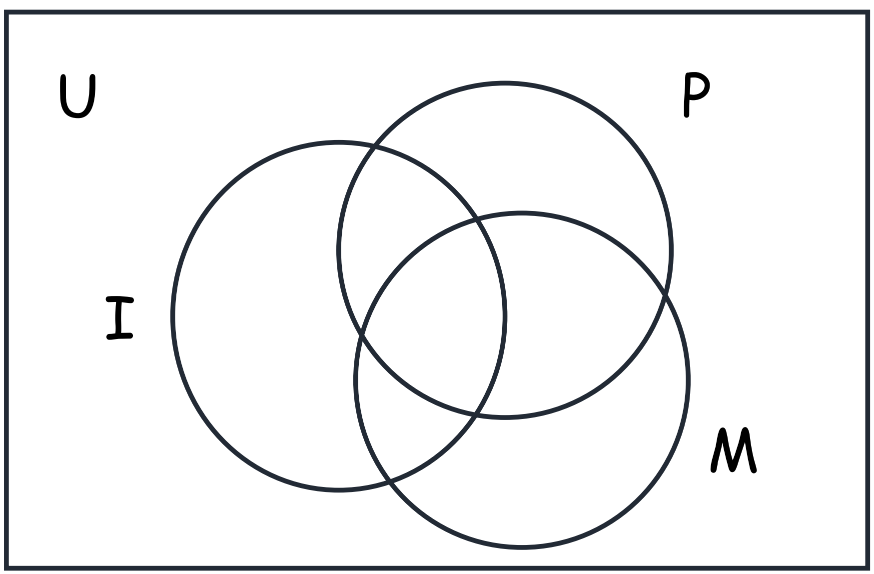
$|I| = 25, |P| = 30, |I \cap P| = 10$

$|\bar{I} \cap \bar{P}| = ?$

$|\bar{I} \cap \bar{P}| = |\overline{I \cup P}| = |U| - |I \cup P|$

Principles of Inclusion & Exclusion

Assume there 80 students enrolled to freshman engineering program this year. 30 of them also enrolled in Physics, 25 of them also enrolled in Introdution to Programming, and 10 of them also enrolled in both Physics and Intoduction to Programming courses.
20 of them taking Math, 5 of them taking both Math and Intro, 15 of them taking Math and Physics, 3 of them taking all. What is the number of students not taking any of them ?



$|U| = 80$

$$|I \cup P \cup M| = |I| + |P| + |M| - |I \cap P| - |I \cap M| - |M \cap P| + |I \cap P \cap M|$$

$|I| = 25,\ |P| = 30,\ |I \cap P| = 10$

$|M| = 20,\ |I \cap M| = 5,\ |P \cap M| = 15$

$|I \cap P \cap M| = 3$

$$|I \cup P \cup M| = 30 + 25 + 20 - 10 - 5 - 15 + 3 = 48$$

$$|\overline{I \cup P \cup M}| = 80 - 48 = 32$$

Principles of Inclusion & Exclusion

- Find the number of positive integers sitrictly less than 101 that is not divisible by 2, 3, and 5 ?

$$A = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 2\}$$
$$B = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 3\}$$
$$C = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 5\}$$

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = ?$$

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C| = ?$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$A \cap B = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 6\}$$
$$A \cap C = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 10\}$$
$$B \cap C = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 15\}$$
$$A \cap B \cap C = \{x \in Z |\ 1 \leq x \leq 100 \text{ and x is divisble by } 30\}$$

$$|A \cup B \cup C| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$
$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 100 - 74 = 26$$

Principles of Inclusion & Exclusion

- In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns 'car', 'dog', 'pun', and 'byte' occurs ?

  . . . efcardxyz . . .   . . . efcarxdogbus . . .

Let's define a set A, that contains all permutations of 26 letters in which the pattern car occurs.
In similar way, B for dog, C for pun, and D for byte

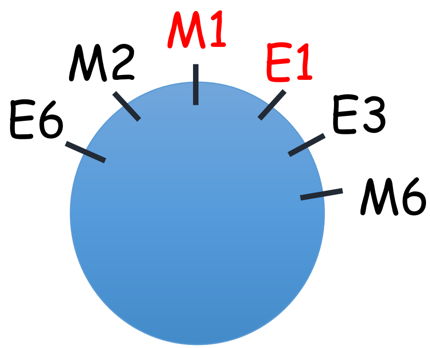
$|A| = 24!,\ |B| = 24!,\ |C| = 24!,\ |D| = 23!$
$|A \cap B| = 22!,\ |A \cap C| = 22!,\ |A \cap D| = 21!,$
$|B \cap C| = 22!,\ |B \cap D| = 21!,\ |C \cap D| = 21!$
$|A \cap B \cap C| = 20!,\ |A \cap B \cap D| = 19!,\ |A \cap C \cap D| = 19!,\ |B \cap C \cap D| = 19!$
$|A \cap B \cap C \cap D| = 17!$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| \\ - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| \\ + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$|A \cup B \cup C \cup D| = 3.24! + 23! - 3.22! - 3.21! + 20! + 3.19! - 17! = K$
$|U| - |A \cup B \cup C \cup D| = 26! - K$

Principles of Inclusion & Exclusion

- Six married couples are seating around a round table. In how many arrangements does no wife sit next to her husband ?



Let's define a set S_i that contains all the arrangements of 6 couples in which the couple i seat next to each other

$S_1 \cup S_2 \cup \ldots \cup S_6$: at least one couple sit together

$U - \{S_1 \cup S_2 \cup \ldots \cup S_6\}$: there is no couple that sit together

$11! - K$

$|S_i| = 2(10!)$ where $i \in [6]$

$|S_i \cap S_j| = 2^2(9!)$ where $i, j \in [6]$

$|S_i \cap S_j \cap S_k| = 2^3(8!)$, where $i, j, k \in [6]$

$|S_{i_1} \cap S_{i_2} \cap S_{i_3} \cap S_{i_4}| = 2^4(7!)$, where $i_j \in [6]$

$|S_{i_1} \cap S_{i_2} \cap S_{i_3} \cap S_{i_4} \cap S_{i_5}| = 2^5(6!)$, where $i_j \in [6]$

$|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6| = 2^6(5!)$

$$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6| = \sum|S_i| - \sum|S_i \cap S_j| + \sum|S_i \cap S_j \cap S_k| - \sum|S_{i_1} \cap S_{i_2} \cap S_{i_3} \cap S_{i_4}| + \sum|S_{i_1} \cap S_{i_2} \cap S_{i_3} \cap S_{i_4} \cap S_{i_5}| - |S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6|$$

$$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6| = \binom{6}{1}2(10!) - \binom{6}{2}2^2(9!) + \binom{6}{3}2^3(8!) - \binom{6}{4}2^4(7!) + \binom{6}{5}2^5(6!) - \binom{6}{6}2^6(5!) = K$$

Principles of Inclusion & Exclusion

- $x_1 + x_2 + x_3 + x_4 = 18$ where $x_i \leq 7$ for $1 \leq i \leq 4$. How many different non-negative integer solution sets are there ?

  (5, 3, 4, 6), (7, 6, 5, 0), (7, 7, 3, 1), . . .

$S_i = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 18,\ x_i > 7\}$ $\quad \binom{13}{10}$
- solve the equation $x_1 + x_2 + x_3 + x_4 = 10$
- then add 8 to x_i in the solution to find the elements of the set S_i

$S_i \cap S_j = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 18,\ x_i, x_j > 7\}$ $\quad \binom{5}{2}$
- solve the equation $x_1 + x_2 + x_3 + x_4 = 2$
- then add 8 to x_i and x_j in the solution to find the elements of the set $S_i \cap S_j$

$S_i \cap S_j \cap S_k = \{(x_1, x_2, x_3, x_4) | x_1 + x_2 + x_3 + x_4 = 18,\ x_i, x_j, x_k > 7\}$

$|S_i \cap S_j \cap S_k| = 0$ and $|S_1 \cap S_2 \cap S_3 \cap S_4| = 0$

$|S_1 \cup S_2 \cup S_3 \cup S_4| = 4\binom{13}{10} - \binom{4}{2}\binom{5}{2}$ and $|U| = \binom{21}{18}$

$$|U| - |S_1 \cup S_2 \cup S_3 \cup S_4| = \binom{21}{18} - 44$$