Discrete Probability

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Probability

- Sample space of an experiment is the set of all possible outcomes.

- An event is a subset of the sample space

Definition : If sample space contains equally likely outcomes (not biased), for an event, then the probability that E happens

$$p(E) = \frac{|E|}{|S|}$$

Probability

- A bag contains 5 blue and 4 green balls. What is the probability that a ball randomly chosen from the bag is blue?

  5/9

- You roll a dice. What is the probability that you get a 5 or 6 ? an odd number ?

  2/6 3/6

- Suppose you roll two fair dice. What will be the probability that the sum is 6 ? the sum is less than 4 ?

  (1, 5), (2, 4), (3, 3), (5, 1), (4, 2) 5/36

  (1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (3, 1) 6/36

Probability

- Suppose you are choosing 5 cards from a standard deck of 52 cards. What is the probability of having three aces and two kings ? three aces and a pair ?

$$\frac{\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} \qquad \frac{\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}}$$

- What is the probability of having 5 different kinds ?

$$\frac{\dfrac{52 \times (52-4) \times (52-4.2) \times (52-4.3) \times (52-4.4)}{5!}}{\binom{52}{5}} \qquad \frac{\binom{13}{5} 4^5}{\binom{52}{5}}$$

Probability

Definition : the probability of the complementary event of an event E, $\bar{E}$: $p(\bar{E}) = 1 - p(E)$

- A sequence of 5 bits randomly generated. What is the probability that at least two bits are 1's ?

E : at most one bit is 1, $p(E) = \frac{\binom{5}{0}+\binom{5}{1}}{2^5} = \frac{6}{32}$

$p(\bar{E}) = 1 - p(E) = 1 - \frac{6}{32} = \frac{26}{32}$

- What is the probability a fair die comes up at least one 2 when it is rolled 5 times ?

E : never comes up 2; (1, 1, 6, 1, 5), (4, 1, 5, 3, 3)

$p(E) = \frac{5^5}{6^5}$, thus $p(\bar{E}) = 1 - \frac{5^5}{6^5}$

Probability

- You toss a fair coin four times. What is the probability you get two heads and two tails ? (order not matters)

These are independent events.

HTTH, THTT, HTTT, HHHH, . . . $p(E) = \frac{6}{16}$

- A positive integer selected from [1,100]. What is the probability that the selected number is divisible by either 3 or 5 ?

These are not independent events.

Definition : Let E_1 and E_2 be events in the sample space. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$|E_1| = 33, |E_2| = 20$, and $|E_1 \cap E_2| = 6$; thus, $p(E_1 \cup E_2) = \frac{33+20-6}{100}$

Probability

- $p(E) = \frac{|E|}{|S|}$; this definition assumes that all outcomes are equally likely (the probability of each element is same). What if we work on a biased sample space.

Definition : Let S be the sample space of an experiment with a finite or countable number of outcomes. A funtion p is called probability distribution that assigns a value to each possible outcome of S. There are two conditions p must satisfy:

- $0 \leq p(x) \leq 1, \forall x \in S$
- $\sum_{x \in S} p(x) = 1$

Probability

- What do we assign to heads and tails, if we deal a fair coin ? if we deal a biased coin so that heads comes up twice as often as tails ?

$p(H) = p(T)$ and $p(H) + p(T) = 1$, so $p(H) = p(T) = 1/2$

$p(H) = 2.p(T)$ and $p(H) + p(T) = 1$, so $p(T) = 1/3$ and $p(H) = 2/3$

- Suppose that a dice is biased so that 5 appears twice as each other number but other five outcomes are equally likely. What is the probability that an odd number appears when we roll the dice ?

$p(1) = p(2) = p(3) = p(4) = p(6)$, $p(5) = 2.p(6)$, and
$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$, so $7.p(6) = 1 \rightarrow p(6) = 1/7$

$p(E) = p(1) + p(3) + p(5) = 4/7$

Probability

- Suppose we flip a fair coin three times, and we know that the event F in which the first flip comes up tails, occurs. Given this information, what is the probability that an odd number of tails appear ?

E : the event that an odd number of tails appear

F ={ THH, THT, TTH, TTT }, p(E) = 2/8

Definition : Let E and F be events such that p(F) > 0. The conditional probability of E given F, denoted by p(E | F), is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

Probability

- Suppose a bit string of length 4 is randomly generated. All ourcomes are equally likely. What is the probability that the string contains at least two consecutive 0's given its first bit is 0 ?

E : the event that the string contains at least two consecutive 0's

F : the event that the first bit of the string is 0

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

$$\mathrm{E} \cap \mathrm{F} = \{0001, 0000, 0010, 0011, 0100\}$$

$$p(E \cap F) = 5/16,\ p(F) = 8/16,\ p(E \mid F) = (\tfrac{5}{16})/(\tfrac{8}{16}) = 5/8$$

Probability

- If E and F are independent events, then the occurence of F (or E) gives no information about the event E (or F). So,

$$p(E \mid F) = p(E)$$

Since $p(E \mid F) = \frac{p(E \cap F)}{p(F)}$,

$$p(E \cap F) = p(E).p(F)$$

- Suppose E is the event that a randomly generated bit string of length 6 begins with 0 and F is the event that this bit string includes an even number of 0's. Are E and F independent if all the outcomes are equally likely in this sample space ?

|E| = 2^5, and |F| = $\binom{6}{0} + \binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 32$ (half of the all)

$$|E \cap F| = 16 \rightarrow p(E \cap F) = \frac{1}{4} = p(E).p(F) = \left(\frac{1}{2}\right).\left(\frac{1}{2}\right) = \frac{1}{4}$$

Probability

- Let *E* and *F* be the events that a family of *4* children has children of both sexes and has at most one boy, respectively. Are *E* and *F* independent ?

$|E| = 2^4 - |\{BBBB, GGGG\}| = 14$

$|F| = 1 + \binom{4}{1} = 5$

$|E \cap F| = \binom{4}{1} = 4$, thus $p(E \cap F) = \frac{1}{4} \neq p(E).p(F) = \left(\frac{14}{16}\right).\left(\frac{5}{16}\right) = \frac{35}{128}$

- The events $E_1, E_2, \ldots, E_n$ are pairwise independent if and only if $p(E_i \cap E_j) = p(E_i).p(E_j)$ for all pairs of integers i and j such that $1 \leq i < j \leq n.$

Bernoulli Trials

Definition : Suppose that an experiment can have only two possible outcomes (flipping a coin, or generating a random bit). Each performance of such experiment is called a Bernoulli Trial.

- The probability of having heads is 2/3 for a biased coin. What is the probability that exactly four heads come up when the coin is flipped seven times ?

TTHHTHH - $\frac{1}{3}.\frac{1}{3}.\frac{2}{3}.\frac{2}{3}.\frac{1}{3}.\frac{2}{3}.\frac{2}{3} = (\frac{2}{3})^4(\frac{1}{3})^3$ for each

$\binom{7}{4}(\frac{2}{3})^4(\frac{1}{3})^3$ for total

The probability of exactly k successes in n independent Bernoulli trials, with the probability of success p and the probability of failure q = 1 – p,

$$\binom{n}{k} p^k q^{n-k}$$

Bernoulli Trials

The probability of exactly k successes in n independent Bernoulli trials, with the probability of success p and the probability of failure
q = 1 – p,

$$\binom{n}{k} p^k q^{n-k}$$

- Suppose that the probability that the bit 0 is generated is 0.9, the probability that the bit 1 is generated is 0.1, and bits are generated independently. What is the probability that exactly eight 0 are generated when 10 bits are randomly sampled ?

p = 0.9, q = 0.1 $\binom{10}{8}(0.9)^8(0.1)^2$

Bayes' Theorem

Assume there are two boxes containing green and red balls. First one contains 2 green and 7 red balls, second one contains 4 green and 3 red balls. Assume you randomly select one box, then randomly select one ball from this box. If the ball you selected is a red ball, What is the probability that you selected that ball from the first box ?

E : you select a red ball
F : you select the ball from the first box
$\bar{E}$: you select a green ball
$\bar{F}$: you select the ball from the second box

$p(F|E) = ?$

$p(F|E) = \frac{p(F \cap E)}{p(E)} \Rightarrow p(E|F) = \frac{p(F \cap E)}{p(F)} \Rightarrow p(F \cap E) = p(E|F)p(F)$

$p(F) = \frac{1}{2}$ and $p(E|F) = \frac{7}{9} \Rightarrow p(F \cap E) = \frac{7}{18}$

$$p(F|E) = \frac{\frac{7}{18}}{\frac{38}{63}} = \frac{49}{76}$$

$E = E \cap U = E \cap (F \cup \bar{F}) = (E \cap F) \cup (E \cap \bar{F})$

$p(E) = p(E \cap F) + p(E \cap \bar{F}) = p(E|F)p(F) + p\left(E\middle|\bar{F}\right)p(\bar{F}) = \frac{7}{9}.\frac{1}{2} + \frac{3}{7}.\frac{1}{2} = \frac{38}{63}$

Bayes' Theorem

Assume there are two boxes containing green and red balls. First one contains 2 green and 7 red balls, second one contains 4 green and 3 red balls. Assume you randomly select one box, then randomly select one ball from this box. If the ball you selected is a red ball, What is the probability that you selected that ball from the first box ?

E : you select a red ball

F

$\bar{E}$

$\bar{F}$

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F) + p(E|\bar{F})p(\bar{F})}$$

$p(F|E) = ?$

$$p(F|E) = \frac{p(F \cap E)}{p(E)} \Rightarrow p(E|F) = \frac{p(F \cap E)}{p(F)} \Rightarrow p(F \cap E) = p(E|F)p(F)$$

$p(F) = \frac{1}{2}$ and $p(E|F) = \frac{7}{9} \Rightarrow p(F \cap E) = \frac{7}{18}$

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$$E = E \cap U = E \cap (F \cup \bar{F}) = (E \cap F) \cup (E \cap \bar{F})$$

$$p(E) = p(E \cap F) + p(E \cap \bar{F}) = p(E|F)p(F) + p(E|\bar{F})p(\bar{F}) = \frac{7}{9}.\frac{1}{2} + \frac{3}{7}.\frac{1}{2} = \frac{38}{63}$$

Bayes' Theorem

Suppose one person in 1000 has a particular disease and there is a fairly accurate diagnostic test for it. 90% of the time this test gives a positive result for the people having this disease, and 95% of the time this test gives a negative result for the people not having this disease.
What is the probability that a person that gets a positive result has the disease?
What is the probability that a person that gets a negative result does not have the disease?

E : a person, randomly selected, has a positive result for the disease
F : a person, randomly selected, has the disease

$$p(F|E) = ? \qquad p(\bar{F}|\bar{E}) = ?$$

$$p(F) = \frac{1}{1000} = 0.001, \; p(\bar{F}) = 0.999$$

$$p(E|F) = 0.9, \; p(\bar{E}|\bar{F}) = 0.95, \; p(\bar{E}|F) = 0.1, \; p(E|\bar{F}) = 0.05$$

$$p(F|E) = \frac{p(E|F)p(F)}{p(E|F)p(F)+p(E|\bar{F})p(\bar{F})} \Rightarrow p(F|E) = \frac{0.9*0.001}{0.9*0.001+0.05*0.999}$$

$$p(\bar{F}|\bar{E}) = \frac{p(\bar{E}|\bar{F})p(\bar{F})}{p(\bar{E}|F)p(F)+p(\bar{E}|\bar{F})p(\bar{F})} \Rightarrow p(\bar{F}|\bar{E}) = \frac{0.95*0.999}{0.95*0.999+0.1*0.001}$$

Bayesian Spam Filters

Suppose there is a set A of spam messages and a set B of non-spam messages. We then identify the words in A and B, and count the number of messages that contains those words.

For a particular word w, let $n_A(w)$ and $n_B(w)$ be the number of messages that contains w in A and B, respectively.

The probability that a spam message contains w is $p(w) = n_A(w)/|A|$, and the probability that a non-spam message contains w is $\mathrm{q}(w) = n_B(w)/|B|$.

For a new e-mail message that contains the word w,

E : the message is spam

F : the message contains the word w

$p(E|F)$: the probability that the message is spam, given that it contains w

$$p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E)+p(F|\bar{E})p(\bar{E})}$$

To simplify the formula, assume that $p(E) = p(\bar{E}) = 1/2$. Thus,

$$p(E|F) = \frac{p(F|E)}{p(F|E)+p(F|\bar{E})} = \frac{p(w)}{p(w)+q(w)}$$

Bayesian Spam Filters

Suppose that the word 'rolex' appears in 125 of 1000 messages which were identified as spam, and in 5 of 1000 messages which were identified as non-spam. Assume that it is equally likely that an incoming message is spam or non-spam. Estimate the probability that an incoming message containing 'rolex' is spam. If our threshold for rejecting a message as spam is 0.9, will such messages be rejected or not ?

E : the message is spam
F : the message contains 'rolex'
$p(E|F)$: the probability that the message is spam, given that it contains 'rolex'

$p(F|E) = \frac{125}{1000} = 0.125$ and $p(F|\bar{E}) = \frac{5}{1000} = 0.005$

Thus,

$$p(E|F) = \frac{p(F|E)}{p(F|E)+p(F|\bar{E})} = \frac{0.125}{0.125+0.005} \approx 0.962$$

Bayesian Spam Filters

Suppose there is a set A of spam messages and a set B of non-spam messages. We then identify the words in A and B, and count the number of messages that contains those words.

Suppose there are two words w_1 and w_2
For a new e-mail message that contains the word w_1 and w_2,

E : the message is spam
F_1: the message contains the word w_1
F_2: the message contains the word w_2 (F_1 and F_2 are independent)
$p(E|F_1 \cap F_2)$: the probability that the message is spam, given that it contains both w_1 and w_2

$$p(E|F_1 \cap F_2) = \frac{p(F_1|E)p(F_2|E)}{p(F_1|E)p(F_2|E)+p(F_1|\bar{E})p(F_2|\bar{E})}$$

Bayesian Spam Filters

Suppose that the word 'stock' appears in 400 of 1000 messages which were identified as spam and in 60 of 1000 messages which were identified as non-spam, and the word 'rolex' appears in 200 of 1000 messages which were identified as spam and in 25 of 1000 messages which were identified as non-spam. Assume that it is equally likely that an incoming message is spam or non-spam. Estimate the probability that an incoming message containing both 'stock' and 'rolex' is spam. If our threshold for rejecting a message as spam is 0.9, will such messages be rejected or not ?

E : the message is spam
F_1: the message contains the word 'stock'
F_2: the message contains the word 'rolex'
$p(E|F_1 \cap F_2)$: : the probability that the message is spam, given that it contains 'stock' and 'rolex'

$$p(E|F_1 \cap F_2) = \frac{p(F_1|E)p(F_2|E)}{p(F_1|E)p(F_2|E) + p(F_1|\bar{E})p(F_2|\bar{E})} = \frac{(0.4)(0.2)}{(0.4)(0.2) + (0.06)(0.025)} \approx 0.93$$

Random Variables

Definition : It is a function from the sample space to the set of real numbers. It assigns a real number to each possible outcome

- Suppose a coin is flipped three times. Let $X(t)$ be the random variable which is the number of heads that appear when t is the outcome.

TTT, TTH, THT, HTT, THH, HTH, HHT, HHH
$X(TTT) = 0$, $X(THT) = 1$, $X(HHT) = 2$, $X(HHH) = 3$,

Definition : The distribution of a random variable X on a sample space S is the set of pairs $(z, p(X = z))$ for all z in $X(S)$ where $p(X = z)$ is the probability that X takes the value z

$P(X = 0) = 1/8$, $P(X = 1) = 3/8$, $P(X = 2) = 3/8$, $P(X = 3) = 1/8$

Random Variables

Definition : The distribution of a random variable X on a sample space S is the set of pairs (z, p(X = z)) for all z in X(S) where p(X = z) is the probability that X takes the value z

- Suppose a pair of dice is rolled. Let X(t) be the random variable which is the sum of the numbers that appear when t is the outcome. What are the probabilities of X = 4 and X = 9 ?

  X((1,3)) = X((3,1)) = X((2,2)) = 4, p(X = 4) = 3/36
  X((6,3)) = X((3,6)) = X((4,5)) = X((5,4)) = 9, p(X = 4) = 4/36

Expected Value

The expected value can be considered as a weighted average of the values of a random variable. The weight is reflected by the probability. It shows the general behaviour of the distribution.

Definition : the expected value (expectation) of the random variable X on the sample space S is

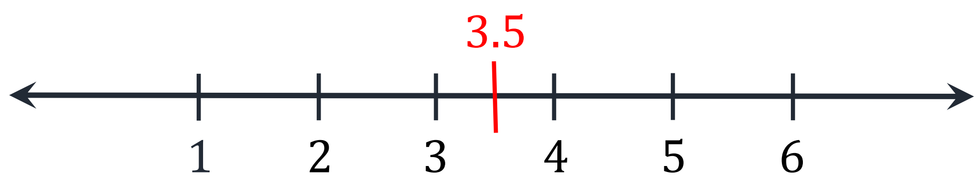
$$E(X) = \sum_{s \in S} p(s)X(s)$$

The deviation of X at s in S : X(s) – E(X), the difference between the value of X and the mean of X

Expected Value

- Let X be the number that comes up when a fair die is rolled. What is the expected value of X ?

$$E(X) = \frac{1}{6}.1 + \frac{1}{6}.2 + \frac{1}{6}.3 + \frac{1}{6}.4 + \frac{1}{6}.5 + \frac{1}{6}.6 = 3.5$$



- A fair coin is flipped 3 times. Let X be the random variable that assigns the number of the heads to each otucome in the sample space. What is the expected value of X ?

TTT, TTH, THT, HTT, THH, HTH, HHT, HHH
X(TTT) = 0, X(THT) = 1, X(HHT) = 2, X(HHH) = 3,

$$E(X) = \frac{1}{8}.0 + \frac{3}{8}.1 + \frac{3}{8}.2 + \frac{1}{8}.3 = 1.5$$

Expected Value

- What is the expected value of the sum of the numbers that appear when a pair of dice is rolled ?

$$E(X) = \frac{1}{36}.2 + \frac{2}{36}.3 + \frac{3}{36}.4 + \frac{4}{36}.5 + \frac{5}{36}.6 + \frac{6}{36}.7 = \frac{2+6+12+20+30+42}{36} = 7$$

- A new employee who is responsible checking the hats of n people at a restaurant, forgets to put claim check numbers on the hats. When customers return for their hats, the checkers gives them back a hat randomly chosen from the remaining hats. What is the expected number of hats that are returned correctly ?

Let X be the random variable that is the number of people receiving the correct hat.
Let X_i be the random variable such that if ith person receives the correct hat $X_i = 1$, otherwise $X_i = 0$.

$$E(X) = E(X_1 + X_1 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

$$E(X_i) = \sum_{s \in S} p(s)X(s) = p(X_i = 1).1 + p(X_i = 1).0 = 1/n$$

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_n) = n.\frac{1}{n} = 1$$

Expected Value

- Suppose that the probability that a coin comes up tails is p This coins is flipped repeatedly until it comes up tails. What is the expected number of flips ?

T, HT, HHT, HHHT, HHHHT

p(T) = p, p(HT) =(1-p)p, p(HHT) = (1-p)(1-p)p, . . .

Let X be the random variable that is the number of flips for an outcome

X(T) = 1, X(HT) = 2, X(HHT) = 3, . . .

$$E(X) = X(T).p(T) + X(HT).p(HT) + X(HHT)p(HHT) + \ldots$$
$$E(X) = 1.p + 2.(1-p)p + 3.(1-p)^2p + \ldots$$
$$E(X) = p[1.(1-p)^0 + 2.(1-p)^1 + 3.(1-p)^2p + \ldots]$$
$$E(X) = p\sum_{i=1}^{\infty} i.(1-p)^{i-1} = p.\frac{1}{p^2} = \frac{1}{p}$$

Expected Value

- Suppose that a casino offers a game for a single player at which a fair cion is tossed. The initial stake is 2 dollars, and it is doubled every time heads appears. The game ends when the first tails appears. What would be the fair price to pay the casino in order to enter the game ?

T, HT, HHT, HHHT, HHHHT, . . .

p(T) = $\frac{1}{2}$, p(HT) = $\frac{1}{2}.\frac{1}{2}$, p(HHT) = $\frac{1}{2}.\frac{1}{2}.\frac{1}{2}$, . . .

Let X be the random variable defined as the amount of Money the player wins

X(T) = 2, X(HT) = 2.2, X(HHT) = 2.2.2, . . .

$$E(X) = X(T).p(T) + X(HT).p(HT) + X(HHT)p(HHT) + \dots$$
$$E(X) = 2.\frac{1}{2} + 2.2.\frac{1}{2}.\frac{1}{2} + 2.2.2.\frac{1}{2}.\frac{1}{2}.\frac{1}{2} + \dots$$
$$E(X) = 1 + 1 + 1 + \dots$$

St. Petersburg Paradox