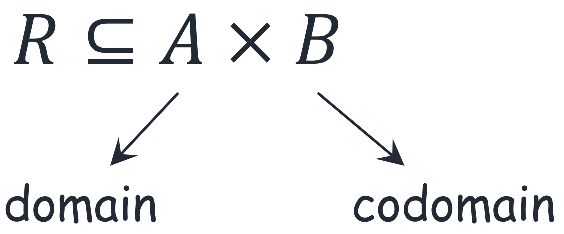
Functions

Murat Osmanoglu

Functions as Relations



$$R \subseteq A \times B$$

domain

codomain

R(A) : the image of R, $R(A) = \{y \in B | (x, y) \in R, \exists x \in A\}$

Function is a relation that satisfies two conditions :

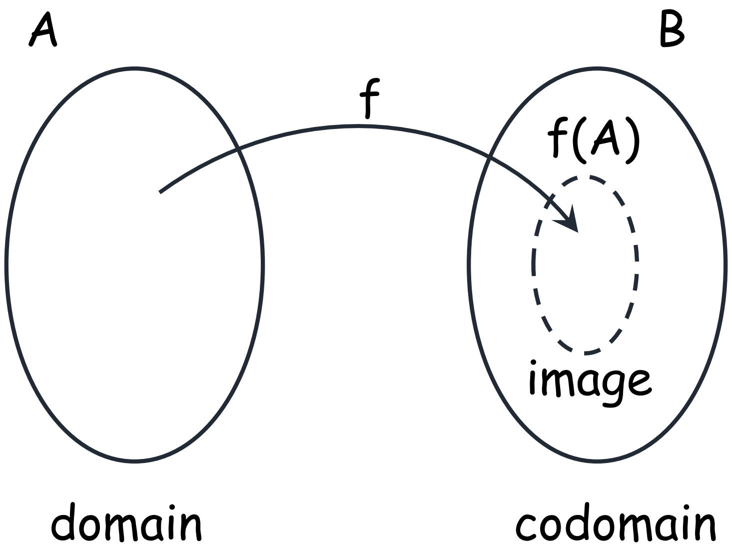
- for every element x of the domain, there is an element y in the range such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[(x \in A) \rightarrow (\exists y \in B \ \ s.t. (x, y) \in R)]$

- for every element x of the domain, there is only one element y of the range such that (x,y) is an element of the relation

  Let $R \subseteq A \times B$ be the relation, $\forall x[((x, y_1) \in R \wedge (x, y_2) \in R) \rightarrow (y_1 = y_2)]$

Definition



- f assigns every element of A to exactly one element of B

$$\text{if } (a,b) \in f\text{, then } f(\underset{\substack{\swarrow\\ \text{preimage}\\ \text{of b}}}{a}) = \underset{\substack{\searrow\\ \text{image}\\ \text{of a}}}{b}$$

Definition

How many functions can be defined from a set A to a set B where |A|=n and |B|=m ?

- Assume $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$

cannot be (a_1, b_1) and (a_1, b_3)

$$f = \{(a_1, \underset{\substack{\uparrow \\ \text{m}}}{\ \ }), (a_2, \underset{\substack{\uparrow \\ \text{m}}}{\ \ }), \ldots, (a_n, \underset{\substack{\uparrow \\ \text{m}}}{\ \ })\}$$

$m^n = |B|^{|A|}$ functions

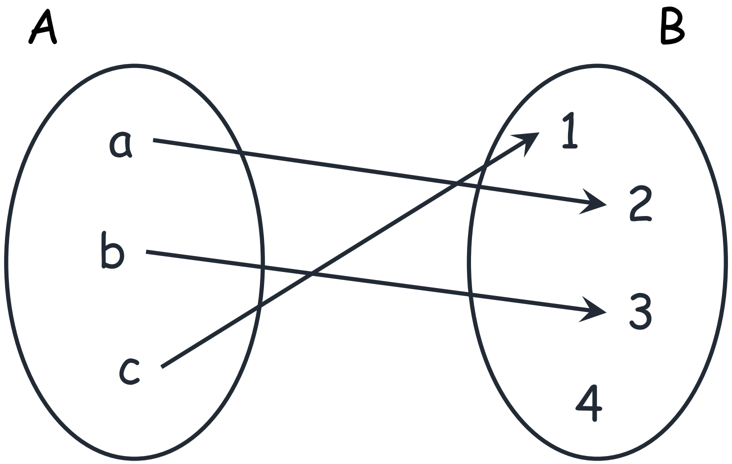
Definition

One-to-One

- Let f : A → B. A function is called one-to-one (or injective) if and only if f(a) = f(b) implies a = b.

$$\forall a \forall b [f(a) = f(b) \rightarrow a = b]$$

or $\forall a \forall b [a \neq b \rightarrow f(a) \neq f(b)]$



Definition

One-to-One

- Let f : A → B. A function is called one-to-one (or injective) if and only if f(a) = f(b) implies a = b.
- Determine whether the function $f(x) = 3x + 1$ ($f: \mathbb{R} \to \mathbb{R}$) is a one-to-one function or not.

$$\forall x_1, x_2 \in \mathbb{R},\ f(x_1) = f(x_2) \to 3x_1 + 1 = 3x_2 + 1$$
$$\to x_1 = x_2$$

- Determine whether the function $f(x) = x^4 - x^2$ ($f: \mathbb{R} \to \mathbb{R}$) is a one-to-one function or not.

$$\forall x_1, x_2 \in \mathbb{R},\ x_1 \neq x_2 \to f(x_1) \neq f(x_2)$$

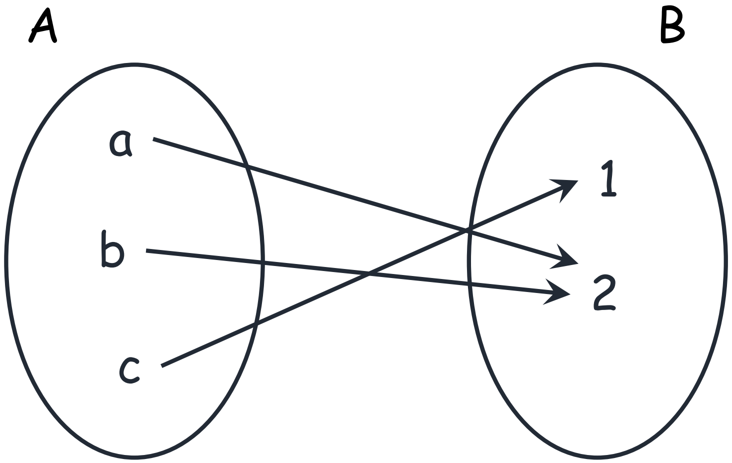
for $x_1 = 1$ and $x_2 = -1$, $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

Definition

Onto

- Let f : A → B. A function is called onto (or surjective) if f(A)=B, i.e. for all $b \in B$, there is at least one $a \in A$ such that f(a) = b

$$\forall b \exists a[f(a) = b]$$



Definition

Onto

- Let f : A → B. A function is called onto (or surjective) if f(A)=B, i.e. for all $b \in B$, there is at least one $a \in A$ such that f(a) = b

- Determine whether the function $f(x) = 3x + 1$ ($f: \mathbb{Q} \rightarrow \mathbb{Q}$) is a onto function or not.

$$\forall b \in \mathbb{Q}, f(a) = b \leftrightarrow 3a + 1 = b$$
$$\leftrightarrow a = \frac{b-1}{3}$$

Since $a = \frac{b-1}{3} \in \mathbb{Q}$, f is onto

- Determine whether the function $f(x) = 3x + 1$ ($f: \mathbb{Z} \rightarrow \mathbb{Z}$) is a onto function or not.

for $5 \in \mathbb{Z}$, there is no integer $x \in \mathbb{Z}$ such that f(x) = 5.

Definition

Bijection

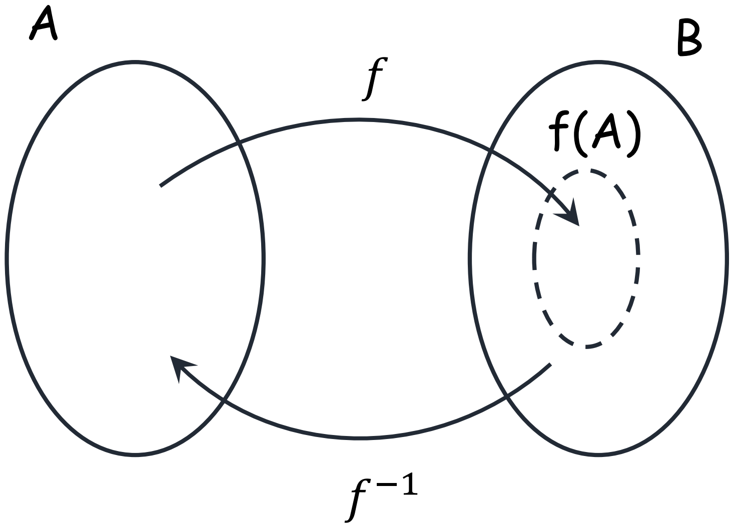
- If a function both one-to-one and onto, it is called bijection.

- the identity function $f(x) = x$ $(f: A \to A)$ is a bijection

  $\forall x_1, x_2 \in A,\ \ f(x_1) = f(x_2) \to x_1 = x_2$

  $\forall a \in A,\ \ f(a) = \mathrm{a}$, the preimage of a is itself

Inverse

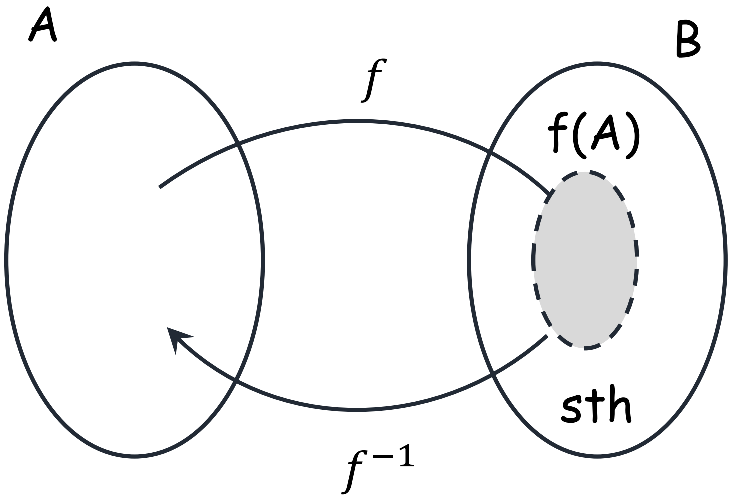


$f: A \to B$

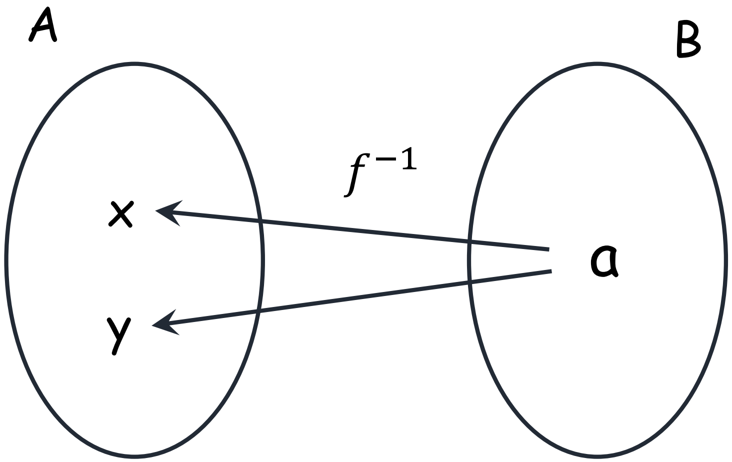
$f(a) = b$

$f^{-1}: B \to A$

$f^{-1}(b) = a$



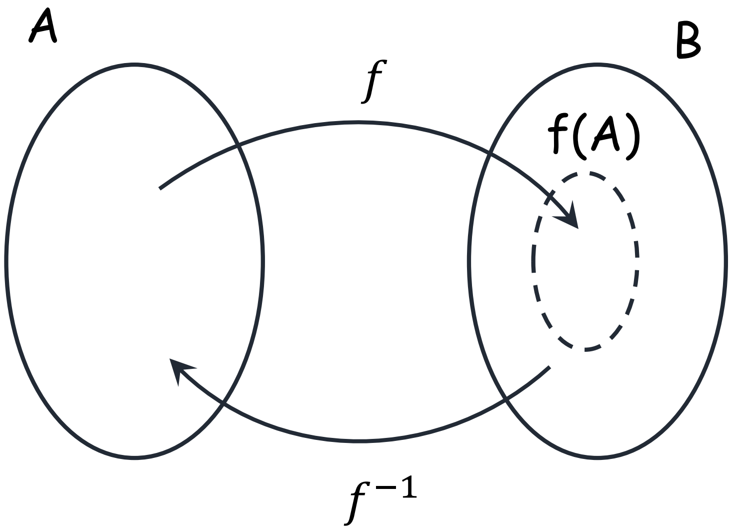
f(A)≠B



$f(x) = f(y) = a$

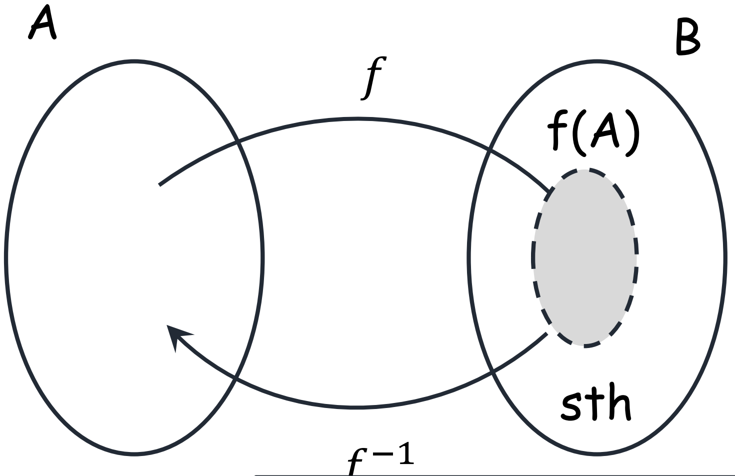
$f^{-1}(a) = x$ and $f^{-1}(a) = y$

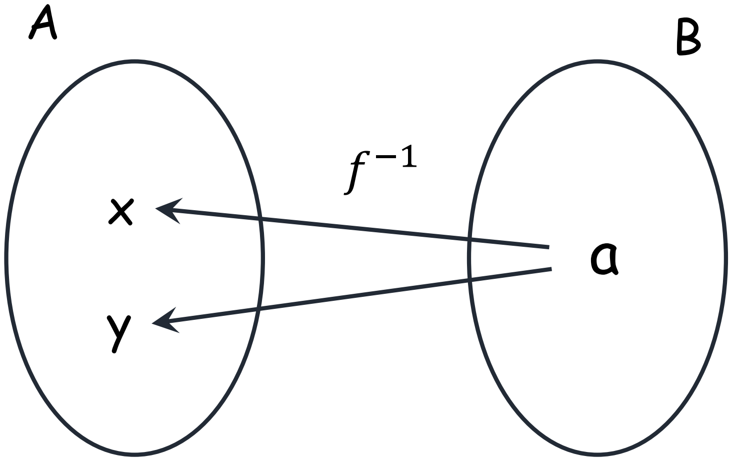
Inverse



$$f: A \to B$$
$$f(a) = b$$

$$f^{-1}: B \to A$$
$$f^{-1}(b) = a$$





If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

$= y$

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \to \mathbb{Z}$, defined as $f(x) = x + 1$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\ f(x_1) = f(x_2) \to x_1 + 1 = x_2 + 1$$

$$\to x_1 = x_2 \text{ (one-to-one)}$$

$$\forall y \in \mathbb{Z}, f(x) = \mathrm{y} \leftrightarrow x + 1 = y$$

$$\leftrightarrow x = y - 1 \in \mathbb{Z} \text{ (onto)}$$

$$f^{-1}(x) = x - 1$$

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f: \mathbb{Z} \rightarrow \mathbb{Z}$, defined as $f(x) = 2x + 1$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\ \ f(x_1) = f(x_2) \rightarrow 2x_1 + 1 = 2x_2 + 1$$

$$\rightarrow x_1 = x_2$$ (one-to-one)

$$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}\ f(x) = \mathrm{y} \leftrightarrow 2x + 1 = y$$

$$\leftrightarrow x = \frac{y-1}{2}$$

but for some $y \in \mathbb{Z}$, $x = \frac{y-1}{2} \notin \mathbb{Z}$ (not onto)

Inverse

- If a function both one-to-one and onto, it is called bijection.

  If f is a bijection, then f^{-1} can be defined, i.e. f is invertible

- $f : \mathbb{Z} \to \mathbb{N}$, defined as $f(x) = \begin{cases} 2x - 1 & if \;\; x > 0 \\ -2x & if \;\; x \leq 0 \end{cases}$, f is invertible ?

$$\forall x_1, x_2 \in \mathbb{Z},\;\; f(x_1) = f(x_2) \to 2x_1 - 1 = 2x_2 - 1$$

$$\to x_1 = x_2$$

$$\forall x_1, x_2 \in \mathbb{Z},\;\; f(x_1) = f(x_2) \to -2x_1 = -2x_2$$

$\to x_1 = x_2$ (one-to-one)

$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$, if $\mathrm{y} = 2k, \exists k \in \mathbb{Z}$, then $f(x) = \mathrm{y} \leftrightarrow -2x = y$

$$\leftrightarrow x = -\frac{y}{2} = -k \in \mathbb{Z}$$

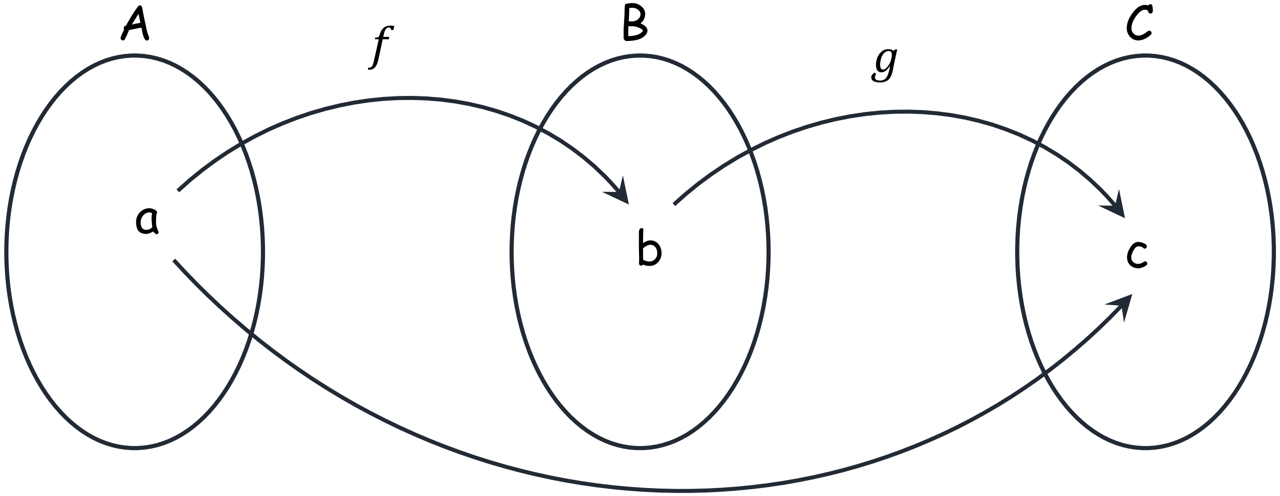
$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}$, if $\mathrm{y} = 2k + 1, \exists k \in \mathbb{Z}$,

then $f(x) = \mathrm{y} \leftrightarrow 2x - 1 = y$

$$\leftrightarrow x = \frac{y+1}{2} = k + 1 \in \mathbb{Z}$$

(onto)

Composition



$g \circ f$

$f\colon A \to B$ and $g\colon B \to C$

$g \circ f\colon A \to C$

$f(a) = b$ and $\mathrm{g}(b) = c$

$g \circ f(a) = g\big(f(a)\big) = g(b) = c$

Composition

- $f, g: \mathbb{Z} \to \mathbb{Z}$,

  $f(x) = 3x + 1$ and $g(x) = 2x - 1$

  $g \circ f(x) = g(f(x)) = g(3x + 1) = 2(3x + 1) - 1 = 6x + 1$

  $f \circ g(x) = f(g(x)) = f(2x - 1) = 3(2x - 1) + 1 = 6x - 2$

- $f: A \to B$

  $f \circ f^{-1}(y) = f(f^{-1}(y) = f(x) = y, \quad f \circ f^{-1} = I_B$

  $f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x, \quad f^{-1} \circ f = I_A$

- If f and g are one-to-one, then $f \circ g$ is also one-to-one.

$\forall x_1, x_2 \in A, f \circ g(x_1) = f \circ g(x_2) \ \to \ f(g(x_1)) = f(g(x_2))$

$\to \ g(x_1) = g(x_2)$ (f is one-to-one)

$\to \ x_1 = x_2$ (g is one-to-one)

Floor and Ceiling Functions

- floor function of a real number x : is the largest integer that is less than or equal to x, denoted by $\lfloor x \rfloor$.

  $\lfloor 1/5 \rfloor = 0$, $\lfloor -1/5 \rfloor = -1$, $\lfloor 3{,}56 \rfloor = 3$, $\lfloor -3{,}56 \rfloor = -4$

  $\lfloor x \rfloor = n$ if $n \le x < n+1$ or $\lfloor x \rfloor = n$ if $x - 1 \le n < x$

- ceiling function of a real number x : is the smallest integer that is greater than or equal to x, denoted by $\lceil x \rceil$.

  $\lceil 1/5 \rceil = 1$, $\lceil -1/5 \rceil = 0$, $\lceil 3{,}56 \rceil = 4$, $\lceil -3{,}56 \rceil = -3$

  $\lceil x \rceil = n$ if $n - 1 < x \le n$ or $\lceil x \rceil = n$ if $x \le n < x + 1$

Floor and Ceiling Functions

- show that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

assume $x = n + \varepsilon$ where n is integer and $0 \leq \varepsilon < 1$

$$0 \leq \varepsilon < \frac{1}{2}$$

$$\lfloor 2n + 2\varepsilon \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$$

$$2n = n + n$$

$$\frac{1}{2} \leq \varepsilon < 1$$

$$\lfloor 2n + 2\varepsilon \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$$

$$2n + 1 = n + n + 1$$

- determine whether $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all $x, y \in \mathbb{R}$.

assume $0 < x, y < \frac{1}{2}$, then $x + y < 1$.

$$\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$$

$$1 \neq 1 + 1$$

Sequences

Definition : A sequence is a function from $\mathbb{N}$ (or $\mathbb{Z}^+$) to a set S, denoted by $\{a_n\}$ where a_n is the general term of the sequence.

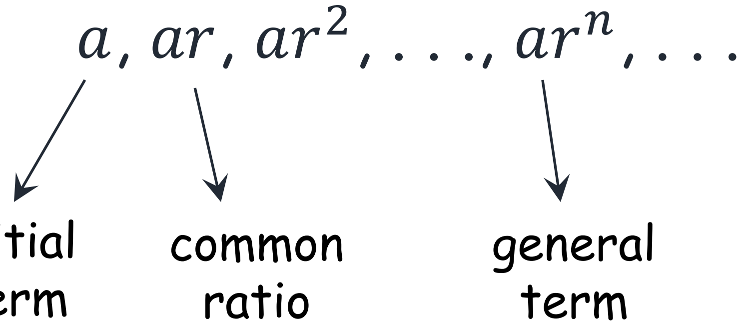
1, 4, 7, 10, 13, . . . $\{3n + 1\}$

0, 1, 3, 7, 15, . . . $\{2^n - 1\}$

- $a_n = \frac{1}{n}$ $a_1 = 1,\ a_2 = \frac{1}{2}, a_3 = \frac{1}{3}, . . .$
- $a_n = \frac{1}{3^n+2}$ $a_0 = \frac{1}{2}, a_1 = \frac{1}{5},\ a_2 = \frac{1}{11}, \ldots$

Sequences

Geometric Sequence :



$$a, ar, ar^2, . . ., ar^n, . . .$$

initial term

common ratio

general term

$$a_n = (-1)^n$$

$$1, -1, 1, -1, \dots$$

$$a_n = 2.3^n$$

$$2, 2.3, 2.9, 2.27, \dots$$

$$a_n = 3.(1/2)^n$$

$$3, 3/2, 3/4, 3/8, \dots$$

Sequences

Arithmetic Sequence :

$$\underset{\substack{\downarrow\\ \text{initial}\\ \text{term}}}{a}, a+\underset{\substack{\downarrow\\ \text{common}\\ \text{difference}}}{d}, a+2d, . . ., \underset{\substack{\downarrow\\ \text{general}\\ \text{term}}}{a+n.d}, . . .$$

$a_n = 1 + n$	$a_n = 2 - 4n$	$a_n = -1 + 8n$
$1, 2, 3, 4, \dots$	$2, -2, -6, -10, \dots$	$-1, 7, 15, 23, \dots$

Summations

- $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_{n-1} + a_n$

  $\sum_{i=0}^{\infty} a_i = a_0 + a_1 + \ldots + a_n + \ldots$

  $\sum_{i=2}^{5} (i^2 - 1) = 4 - 1 + 9 - 1 + 16 - 1 + 25 - 1 = 50$

- $S = \{2, 3, 4\}, \quad \sum_{x \in S} x^3 = 2^3 + 3^3 + 4^3 = 99$

- $\sum cf(x) = c \sum f(x)$

  $\sum (f(x) + g(x)) = \sum f(x) + \sum g(x)$

  $\sum_{i=m}^{n} f(i) = \sum_{i=m}^{k} f(i) + \sum_{i=k+1}^{n} f(i)$

- $$\begin{aligned} \sum_{i=1}^{n} i &= 1 + 2 + \ldots + \frac{n}{2} + \left(\frac{n}{2} + 1\right) + \ldots + (n - 1) + n \\ &= (\mathrm{n} + 1) + (\mathrm{n} + 1) + \ldots + (n + 1) \\ &= \frac{n}{2}(\mathrm{n} + 1) \end{aligned}$$

Summations

- $a, a+d, a+2d, \ldots, a+n.d$

$$\sum_{i=0}^{n}(a+id) = \sum_{i=0}^{n} a + \sum_{i=0}^{n} id$$

$$= \sum_{i=0}^{n} a + d\sum_{i=0}^{n} i$$

$$= (n+1)a + d\,\frac{n(n+1)}{2}$$

- $a, ar, ar^2, \ldots, ar^n$

$$S_n = \sum_{i=0}^{n} ar^i \rightarrow rS_n = r\sum_{i=0}^{n} ar^i = \sum_{i=0}^{n} ar^{i+1}$$

$$rS_n = \sum_{i=1}^{n+1} ar^i = \sum_{i=1}^{n} ar^i + ar^{n+1}$$

$$rS_n = \sum_{i=0}^{n} ar^i + ar^{n+1} - a$$

$$rS_n = S_n + ar^{n+1} - a \quad \rightarrow \quad S_n = \frac{ar^{n+1}-a}{r-1}$$

Recurrence Relations

- sometimes the elements of the sequence are defined recursively in terms of previous and the initial elements of the sequence

$$a_0 = 1, a_1 = 5, a_2 = 13, a_3 = 29, a_4 = ?$$

$$a_1 = 2a_0 + 3 = 5$$
$$a_2 = 2a_1 + 3 = 13$$
$$a_3 = 2a_2 + 3 = 29$$
$$a_4 = 2a_3 + 3 = 61$$

Definition : an equation that express the general term of the sequence in terms of previous terms. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Recurrence Relations

- $a_{n+1} = 3a_n, \ a_0 = 5$

$a_1 = 15 = 3.5$
$a_2 = 75 = 3.(3.5)$
$a_3 = 225 = 3.(3.(3.5))$
$\vdots$
$a_n = 3^n 5$; the unique solution of the given recurrence relation

- $a_{n+1} = d.a_n, \ a_0 = A$ where d is constant

the solution of the recurrence relation will be $a_n = A.d^n$

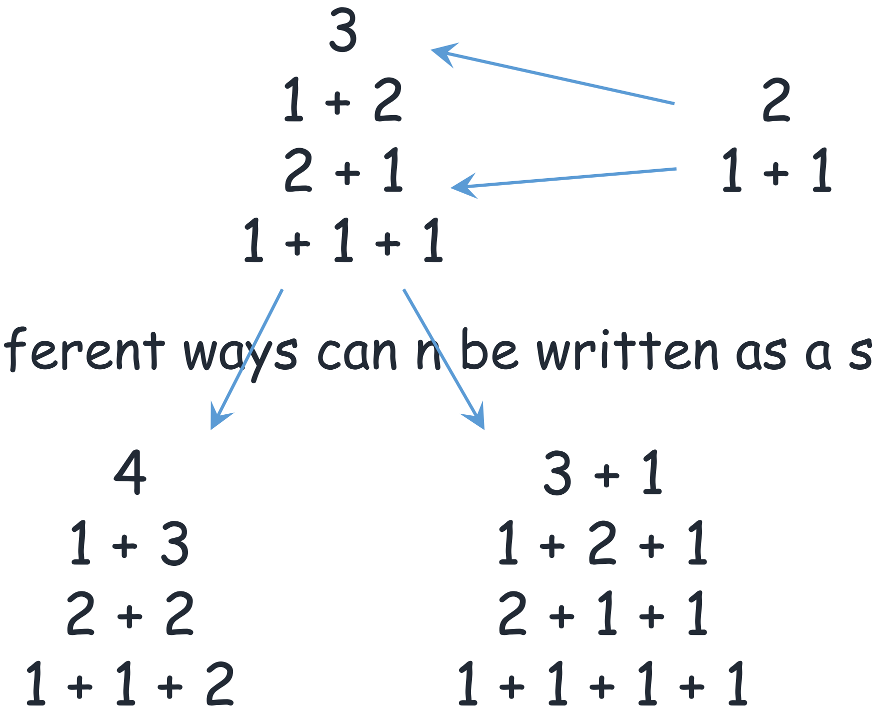
- solve the recurrence relation $a_{n+1} = 7.a_n$ where $n \geq 1$ and $a_2 = 98$

$a_2 = A.7^2 \rightarrow 98 = A.49 \rightarrow A = 2$

the solution is $a_n = 2.7^n$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:



3
1 + 2
2 + 1
1 + 1 + 1

2
1 + 1

- In how many different ways can n be written as a sum of positive integers ?

4
1 + 3
2 + 2
1 + 1 + 2

3 + 1
1 + 2 + 1
2 + 1 + 1
1 + 1 + 1 + 1

- $a_4 = 2. a_3$, $a_3 = 2. a_2$, and $a_2 = 2$

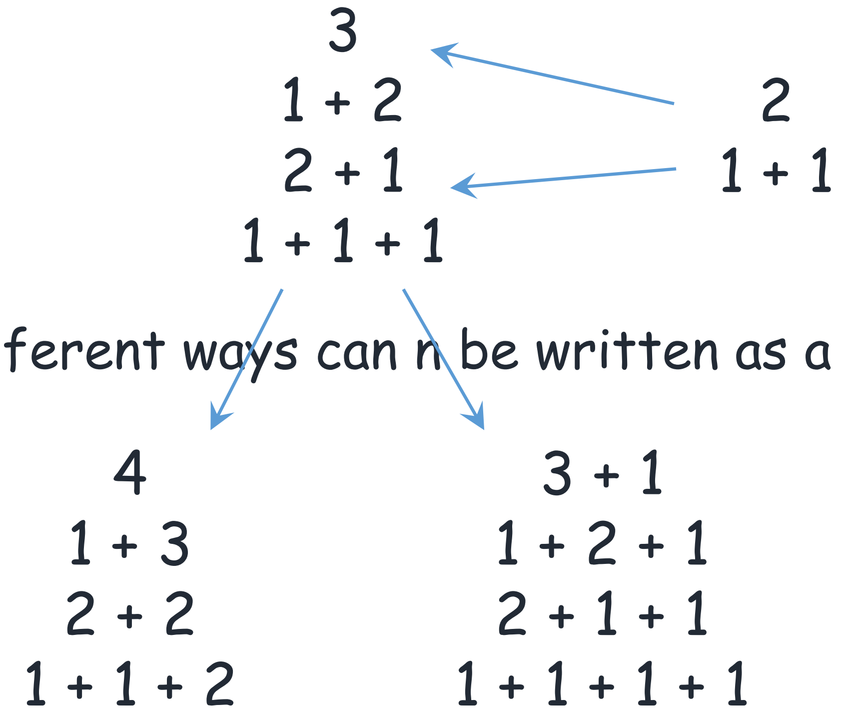
  $a_{n+1} = 2. a_n$, $a_1 = 1$

  create a new sequence $b_n = a_{n+1}$

  $b_n = 2b_{n-1}$, $b_0 = 1$; the solution will be $b_n = 2^n$; thus $a_n = 2^{n-1}$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:



3
1 + 2
2 + 1
1 + 1 + 1

2
1 + 1

- In how many different ways can n be written as a sum of positive integers ?

4
1 + 3
2 + 2
1 + 1 + 2

3 + 1
1 + 2 + 1
2 + 1 + 1
1 + 1 + 1 + 1

- $a_4 = 2.a_3$, $a_3 = 2.a_2$, and $a_2 = 2$

$a_{n+1} = 2.a_n$, $a_1 = 1$

create a new sequence $b_n =$ $_{n+1}$

$b_n = 2b_{n-1}$, $b_0 = 1$; the solution will be $b_n = 2^n$; thus $a_n = 2^{n-1}$

first order linear homogeneous recurrence relation

Recurrence Relations

- $a_{n+1} - d.a_n = 0,\ \ a_0 = A$ where d is constant.
  - first order since a_{n+1} only depends on a_n (the previous term)
  - linear since each variable appears in the first power and there is no product such as $a_{n+1}.a_n$
  - homogeneous since the right hand side is 0
- The second order linear homogeneous recurrence relation :

$$C_0 a_{n+1} + C_1 a_n + C_2 a_{n-1} = 0, a_0 = A, a_1 = B,\ \ n \geq 2$$

- The Fibonacci sequence:

$$F_{n+1} = F_n + F_{n-1},\ \ F_0 = 1,\ \ F_2 = 1,\ \ n \geq 2$$

Recurrence Relations

- The second order linear homogeneous recurrence relation :

$$C_0 a_{n+1} + C_1 a_n + C_2 a_{n-1} = 0,\, a_0 = A,\, a_1 = B,\;\; n \geq 2$$

$a_{n+1} - d.a_n = 0$, $a_0 = A$. the solution was in the form of $a_n = A.d^n$

- Similarly, we look for a solution in the form of $a_n = c.r^n$

If we place it in the equation:

$$C_0 c.r^{n+1} + C_1 c.r^n + C_2 c.r^{n-1} = 0$$

$$C_0 r^2 + C_1 r + C_2 = 0 \quad \text{(characteristic equation)}$$

The solutions for the characteristic equation are called characteristic roots; r_1 and r_2

Recurrence Relations

- $a_{n+1} + a_n - 6a_{n-1} = 0, a_0 = -1, a_1 = 8, \ n \geq 2$

$r^2 + r \ - 6 = 0$ (characteristic equation)

$r_1 = 2, r_2 = -3$ (characteristic roots)

the solution will be in the form of $a_n = c_1 2^n + c_2(-3)^n$.

$a_0 = c_1 2^0 + c_2(-3)^0 \ \rightarrow \ -1 = c_1 + c_2$

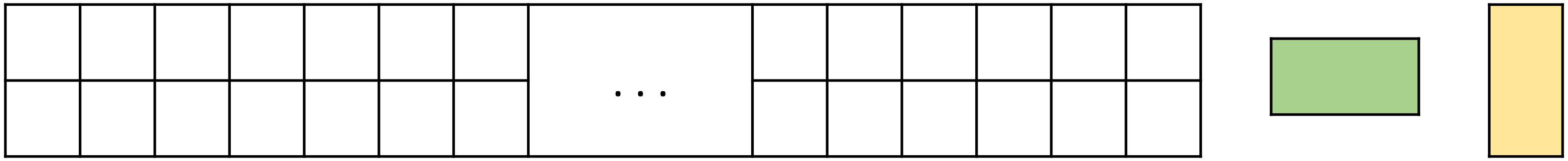
$a_1 = c_1 2^1 + c_2(-3)^1 \ \rightarrow \ 8 = 2c_1 - 3c_2$

$$\begin{array}{r} c_1 + c_2 = -1 \\ 2c_1 - 3c_2 = 8 \\ \hline \end{array} \longrightarrow \quad a_n = 2^n - 2.(-3)^n$$

$c_1 = 1, c_2 = -2$

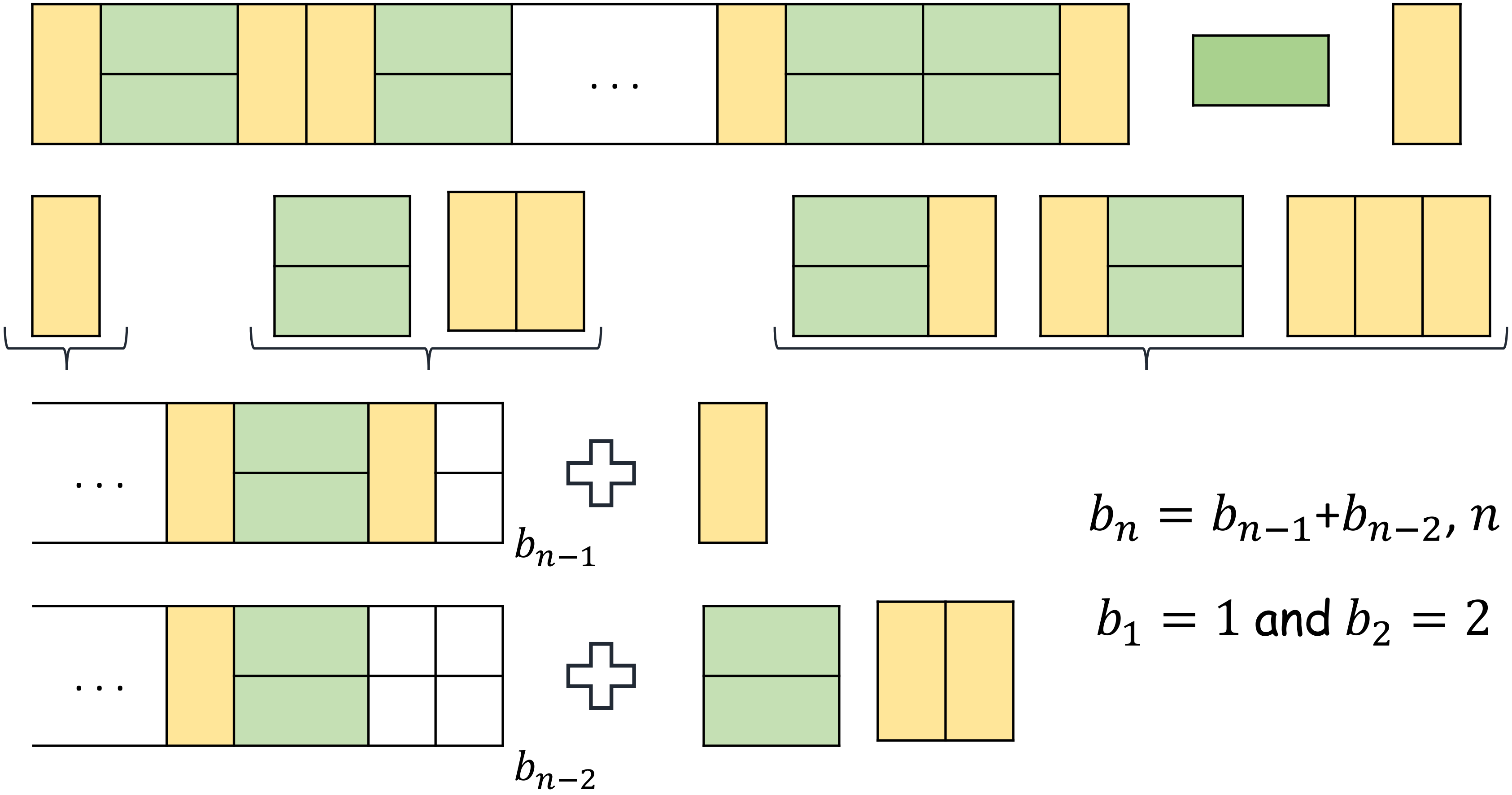
Recurrence Relations

- Suppose we have a 2xn chessboard and we wish to cover it using 2x1 and 1x2 dominoes. In how many different ways can we cover it ?



Recurrence Relations

- Suppose we have a 2xn chessboard and we wish to cover it using 2x1 and 1x2 dominoes. In how many different ways can we cover it ?



$$b_n = b_{n-1}+b_{n-2}, n \geq 3$$

$b_1 = 1$ and $b_2 = 2$

Recurrence Relations

- Suppose we have a 2xn chessboard and we wish to cover it using 2x1 and 1x2 dominoes. In how many different ways can we cover it ?

- $b_n = b_{n-1} + b_{n-2}, n \geq 3, b_1 = 1$ and $b_2 = 2$

$$r^2 - r - 1 = 0$$ (characteristic equation)

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$$ (characteristic roots)

the solution will be in the form of $b_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$

$$b_0 = c_1(\frac{1+\sqrt{5}}{2})^0 + c_2(\frac{1-\sqrt{5}}{2})^0 \rightarrow \quad 1 = c_1 + c_2$$

$$b_1 = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1 \rightarrow \quad 2 = \left(\frac{1+\sqrt{5}}{2}\right)c_1 + (\frac{1-\sqrt{5}}{2})c_2$$

$$c_1 = 1/\sqrt{5}, c_2 = -1/\sqrt{5} \quad \longrightarrow \quad b_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right)$$

Recurrence Relations

- 3 can be written as a sum of positive integers in 4 different ways:

3

1 + 2

2 + 1

1 + 1 + 1

palindrome

they are read
the same from
left to right,
right to left

- How many different palindromes can be found for a given $n \in \mathbb{Z}^+$?

$b_n = 2b_{n-2}, n \geq 3, b_1 = 1$ and $b_2 = 2$

$r^2 - 2 = 0$ (characteristic equation)

$r_1 = \sqrt{2}, r_2 = -\sqrt{2}$ (characteristic roots)

the solution will be in the form of $b_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$

$$b_0 = c_1(\sqrt{2})^0 + c_2(-\sqrt{2})^0 \rightarrow \quad 1 = c_1 + c_2$$

$$b_1 = c_1(\sqrt{2})^1 + c_2(-\sqrt{2})^1 \rightarrow \quad 2 = (\sqrt{2})c_1 + (-\sqrt{2})c_2$$

$$b_n = (\frac{1}{2} + \frac{1}{2\sqrt{2}})(\sqrt{2})^n + (\frac{1}{2} - \frac{1}{2\sqrt{2}})(-\sqrt{2})^n$$