- Ankara Üniversitesi BLM bölümü

BLM433 Sayısal Analiz Teknikleri

Roots of Polynomials

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Follow these rules:

For an nth order equation, there are n real or complex roots.

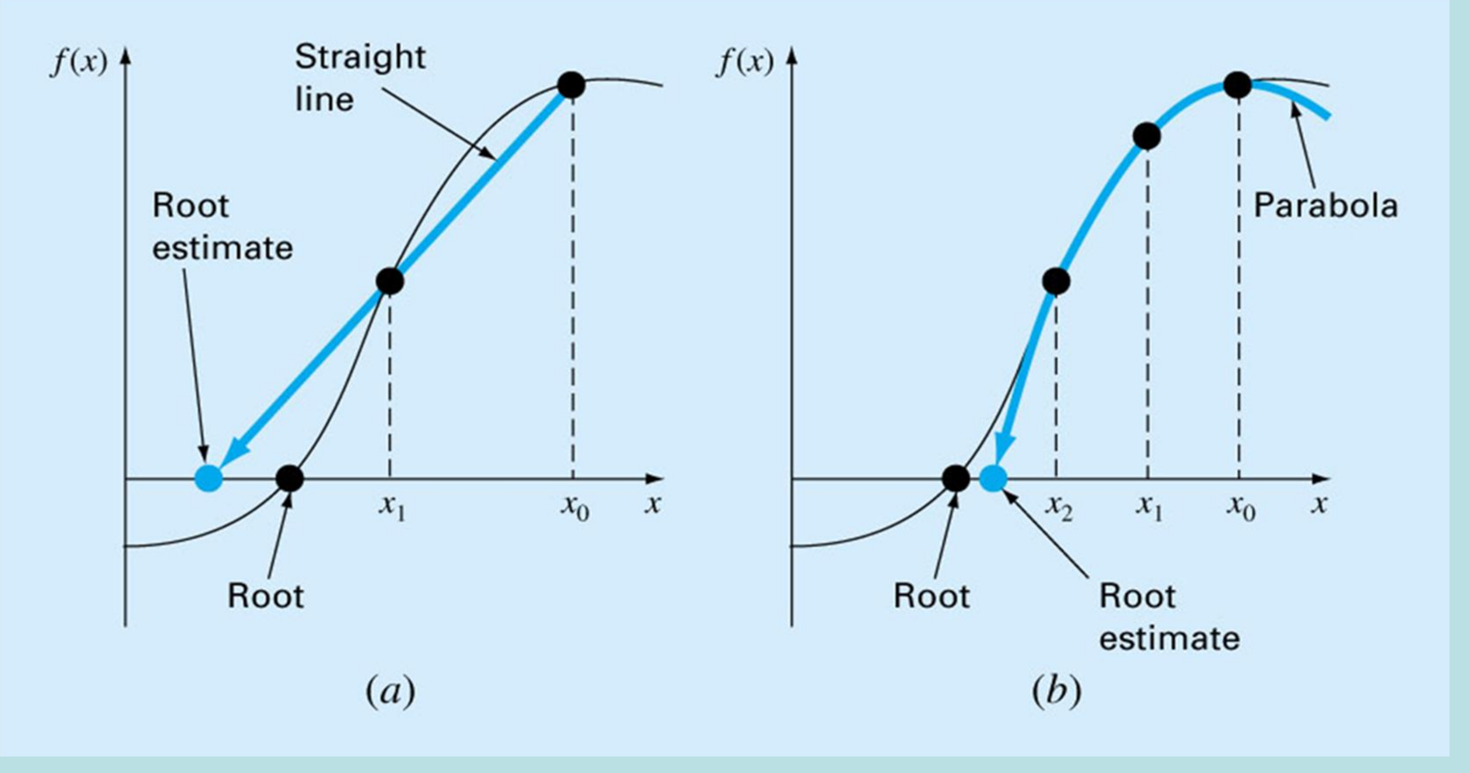
If n is odd, there is at least one real root.

If complex root exist in conjugate pairs (that is, l+mi and l-mi), where i=sqrt(-1).

The efficacy of bracketing and open methods depends on whether the problem being solved involves complex roots. If only real roots exist, these methods could be used. However, Finding good initial guesses complicates both the open and bracketing methods, also the open methods could be susceptible to divergence.

Special methods have been developed to find the real and complex roots of polynomials – Müller and Bairstow methods.

Müller Method



$$f_2(x) = a(x - x_2)^2 + b(x - x_2) + c$$

$$f(x_o) = a(x_o - x_2)^2 + b(x_o - x_2) + c$$

$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$

$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

$$f(x_o) - f(x_2) = a(x_o - x_2)^2 + b(x_o - x_2)$$

$$f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2)$$

If

$$h_o = x_1 - x_o \qquad h_1 = x_2 - x_1$$

$$\delta_o = \frac{f(x_1) - f(x_o)}{x_1 - x_o} \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$(h_o + h_1)b - (h_o + h_1)^2 a = h_o\delta_o + h_1\delta_1$$

$$h_1 b - h_1^2 a = h_1\delta_1$$

$$a = \frac{\delta_1 - \delta_o}{h_1 + h_o} \qquad b = ah_1 + \delta_1 \qquad c = f(x_2)$$

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

$$\varepsilon_a = \left| \frac{x_3 - x_2}{x_3} \right| 100\%$$

- Once x3 is determined, the process is repeated using the following guidelines:
- If only real roots are being located, choose the two original points that are nearest the new root estimate, x3.
- If both real and complex roots are estimated, employ a sequential approach just like in secant method, x1, x2, and x3 to replace xo, x1, and x2.

Bairstow method

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$f_{n-1}(x) = b_1 + b_2 x + b_3 x^2 + \ldots + b_n x^n$$

with a reminder $\mathrm{R} = \mathrm{b}_0$, the coefficien ts are calculated by recurrence relationsh ip

$$b_n = a_n$$

$$b_i = a_i + b_{i+1} t \quad i = n - 1 \text{ to } 2$$

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$f_{n-1}(x) = b_1 + b_2 x + b_3 x^2 + \ldots + b_n x^n$$

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$$b_n = a_n$$

$$b_i = a_i + b_{i+1} t \quad i = n - 1 \text{ to } 2$$

$$f_{n-2}(x) = b_2 + b_3 x + \ldots + b_{n-1} x^{n-3} + b_n x^{n-2}$$

$$R = b_1 (x - r) + b_o$$

Using a simple recurrence relationship

$$b_n = a_n$$

$$b_{n\text{-}1} = a_{n\text{-}1} + r b_n$$

$$b_i = a_i + r b_{i+1} + s b_{i+2} \qquad i = n\text{-}2 \; to \; 0$$

For the remainder to be zero, bo and b1 must be zero. However, it is unlikely that our initial guesses at the values of r and s will lead to this result, a systematic approach can be used to modify our guesses so that bo and b1 approach to zero.

Using a similar approach to Newton Raphson method, both bo and b1 can be expanded as function of both r and s in Taylor series

$$b_1(r+\Delta r, s+\Delta s) = b_1 + \frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s$$

$$b_o(r+\Delta r, s+\Delta s) = b_o + \frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s$$

assuming that the initial guesses are adequately close to the values of r and s at roots. The changes in Δs and Δr needed to improve our guesses will be estimated as

$$\frac{\partial b_1}{\partial r}\Delta r + \frac{\partial b_1}{\partial s}\Delta s = -b_1$$

$$\frac{\partial b_o}{\partial r}\Delta r + \frac{\partial b_o}{\partial s}\Delta s = -b_o$$

$$c_n = b_n$$

$$c_{n-1} = b_{n-1} + rc_n$$

$$c_i = b_i + rc_{i+1} + sc_{i+2} \quad i = n - 2 \text{ to } 2$$

where

$$\frac{\partial b_o}{\partial r} = c_1 \quad \frac{\partial b_o}{\partial s} = \frac{\partial b_1}{\partial r} = c_2 \quad \frac{\partial b_1}{\partial s} = c_3$$

$$c_2 \Delta r + c_3 \Delta s = -b_1$$

$$c_1 \Delta r + c_2 \Delta s = -b_o$$

$$\left|\varepsilon_{a,r}\right| = \left|\frac{\Delta r}{r}\right| 100\%$$

$$\left|\varepsilon_{a,s}\right| = \left|\frac{\Delta r}{r}\right| 100\%$$

$$x = \frac{r \pm \sqrt{r^2 + 4s}}{2}$$

At this point three possibilities exist:

The quotient is a third-order polynomial or greater. The previous values of r and s serve as initial guesses and Bairstow's method is applied to the quotient to evaluate new r and s values.

The quotient is quadratic. The remaining two roots are evaluated directly, using the above eqn.

The quotient is a 1st order polynomial. The remaining single root can be evaluated simply as x=-s/r.