- Ankara Üniversitesi BLM bölümü

BLM433 Sayısal Analiz Teknikleri

- LU decomposition method

Provides an efficient way to compute matrix inverse by separating the time consuming elimination of the Matrix [A] from manipulations of the right-hand side {B}.

Gauss elimination, in which the forward elimination comprises the bulk of the computational effort, can be implemented as an LU decomposition.

If

L- lower triangular matrix

U- upper triangular matrix

Then,

[A]{X}={B} can be decomposed into two matrices [L] and [U] such that

[L][U]=[A]

[L][U]{X}={B}

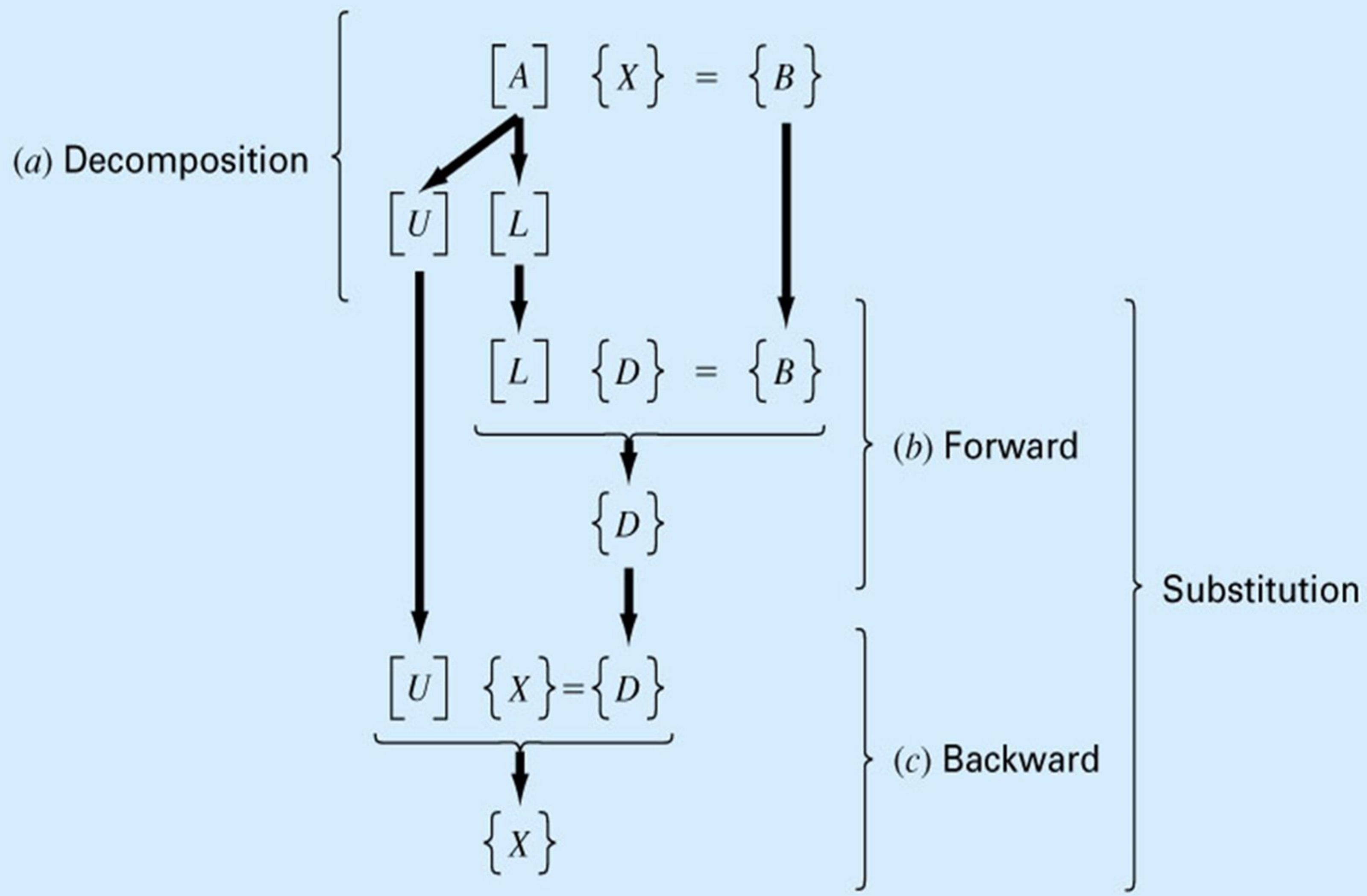
Similar to first phase of Gauss elimination, consider

[U]{X}={D}

[L]{D}={B}

[L]{D}={B} is used to generate an intermediate vector {D} by forward substitution

Then, [U]{X}={D} is used to get {X} by back substitution


[A] {X} = {B}
(a) Decomposition
[U] [L]
[L] {D} = {B}
(b) Forward
{D}
Substitution
[U] {X}={D}
(c) Backward
{X}

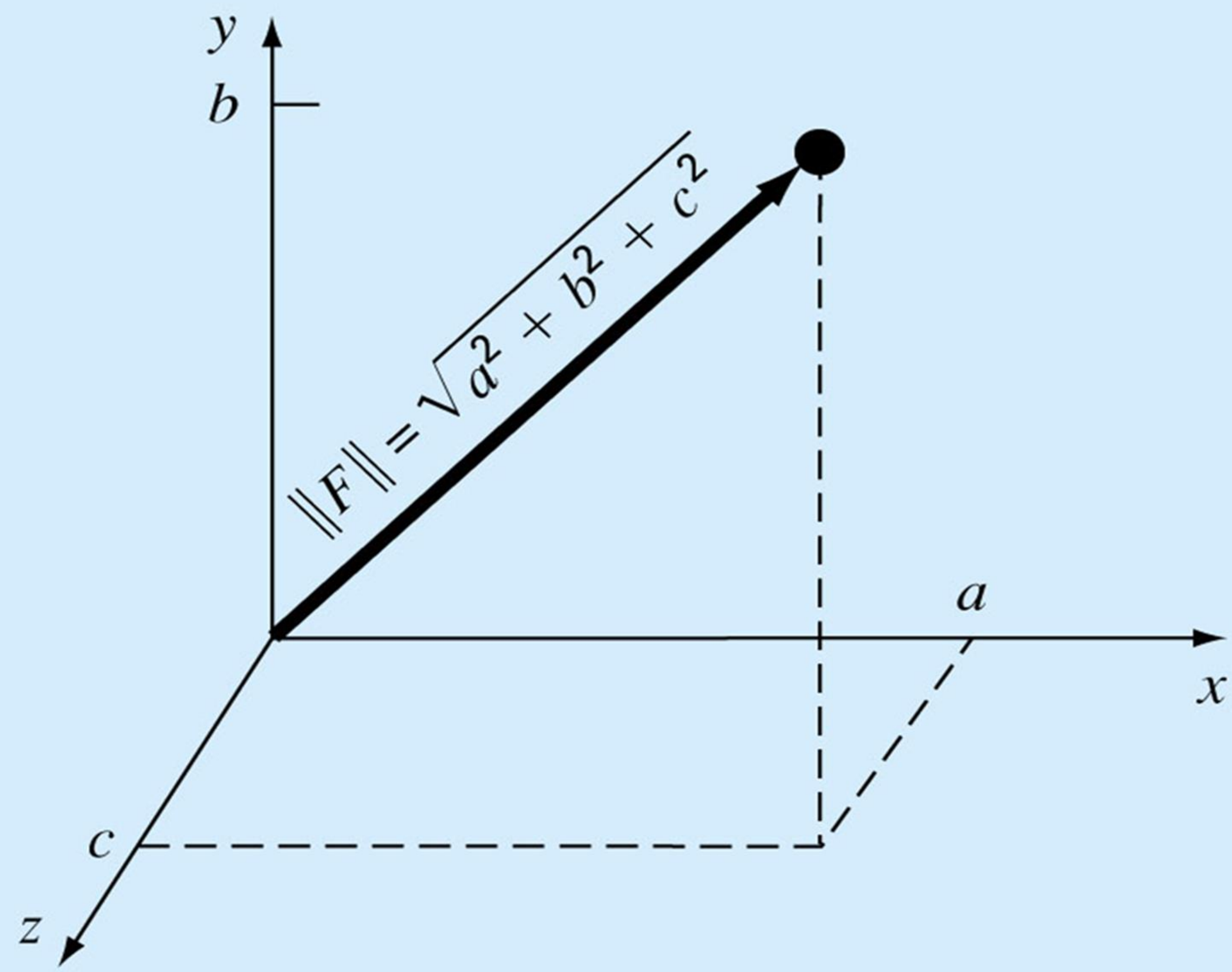
LU decomposition
requires the same total FLOPS as for Gauss elimination.

Saves computing time by separating time-consuming elimination step from the manipulations of the right hand side.

Provides efficient means to compute the matrix inverse

Vector and Matrix Norms

Norm is a real-valued function that provides a measure of size or “length” of vectors and matrices. Norms are useful in studying the error behavior of algorithms.


y
b
$\|F\| = \sqrt{a^2 + b^2 + c^2}$
a
x
c
z

$$\lfloor X \rfloor = \lfloor x_1 \; x_2 \cdots x_n \rfloor$$

a Euclidean norm is computed as

$$\|X\|_e = \sqrt{\sum_{i=1}^{n} x_i^2}$$

For a matrix [A]

$$\|A\|_e = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}^2}$$

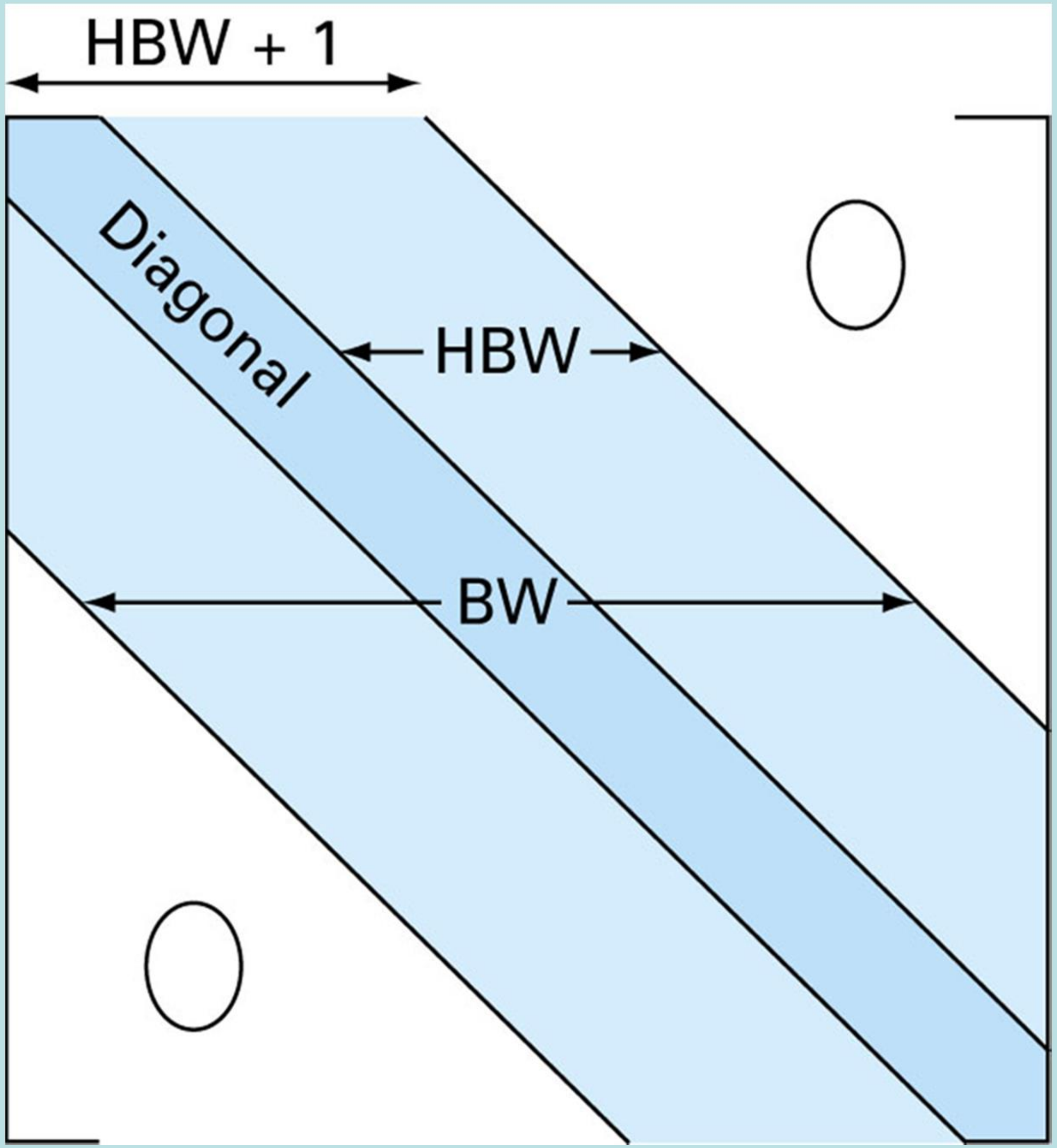
Matrix Condition Number

$$Cond[A] = \|A\| \bullet \|A^{-1}\|$$

$$\frac{\|\Delta X\|}{\|X\|} \le Cond[A] \frac{\|\Delta A\|}{\|A\|}$$

That is, the relative error of the norm of the computed solution can be as large as the relative error of the norm of the coefficients of [A] multiplied by the condition number.

For example, if the coefficients of [A] are known to t-digit precision (rounding errors~10-t) and Cond [A]=10c, the solution [X] matrix


HBW + 1
Diagonal
HBW
BW
0
0

Tridiagonal Systems

$$\begin{bmatrix} f_1 & g_1 & & \\ e_2 & f_2 & g_2 & \\ & e_3 & f_3 & g_3 \\ & & e_4 & f_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

Gauss-Seidel method

Iterative or approximate methods provide an alternative to the elimination methods. The Gauss-Seidel method is the most commonly used iterative method.

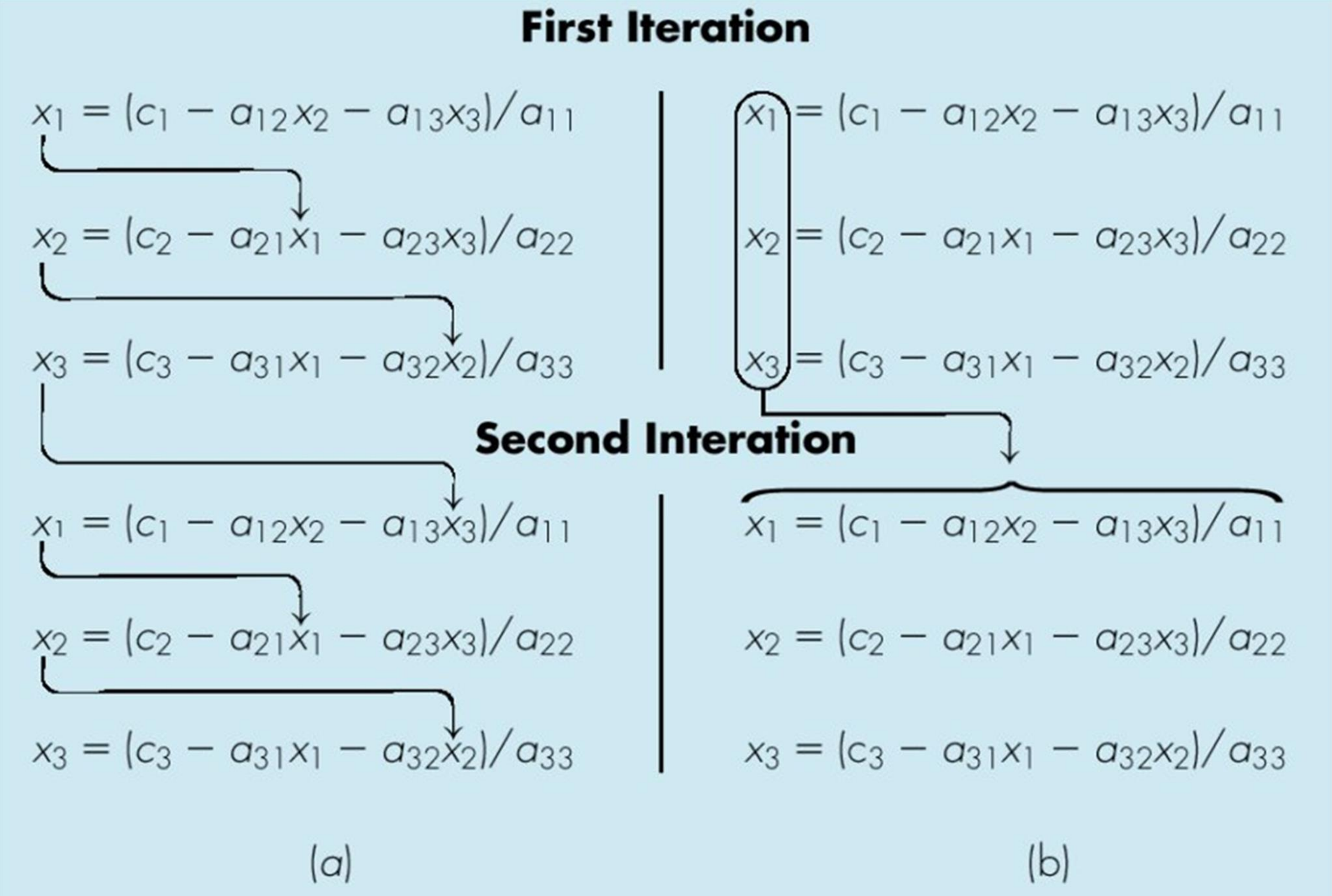
The system [A]{X}={B} is reshaped by solving the first equation for x1, the second equation for x2, and the third for x3, …and nth equation for xn. For conciseness, we will limit ourselves to a 3x3 set of equations.

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$x_1 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| 100\% \prec \varepsilon_s$$


First Iteration
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
Second Interation
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
$x_1 = (c_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$
$x_2 = (c_2 - a_{21}x_1 - a_{23}x_3)/a_{22}$
$x_3 = (c_3 - a_{31}x_1 - a_{32}x_2)/a_{33}$
(a)
(b)

$$\left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| \prec 1$$

$$\left|\frac{\partial v}{\partial x}\right| + \left|\frac{\partial v}{\partial y}\right| \prec 1$$

$$u(x_1, x_2) = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2$$

$$v(x_1, x_2) = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$$

$$\frac{\partial u}{\partial x_1} = 0 \qquad \frac{\partial u}{\partial x_2} = -\frac{a_{12}}{a_{11}}$$

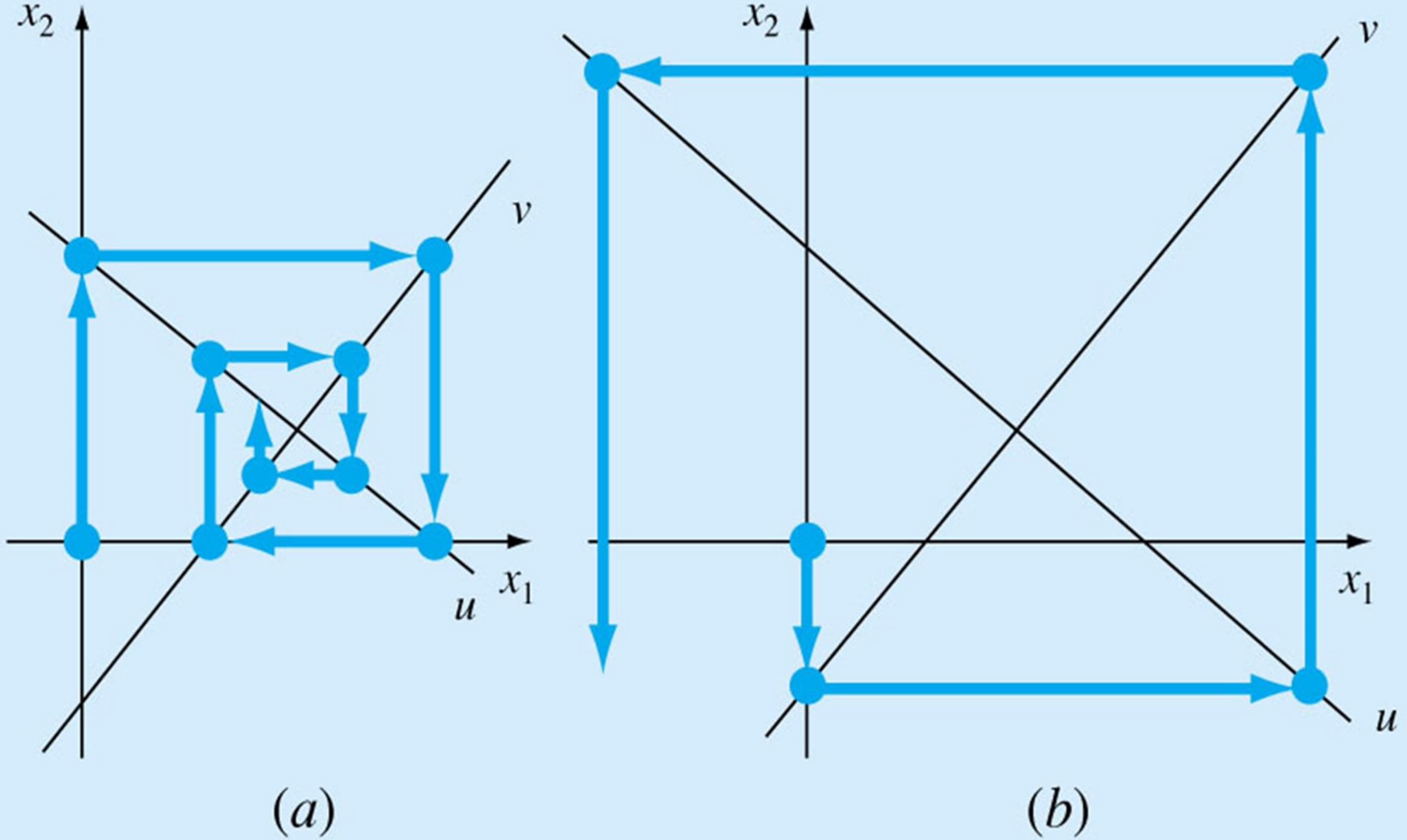
$$\frac{\partial v}{\partial x_1} = -\frac{a_{21}}{a_{22}} \qquad \frac{\partial v}{\partial x_2} = 0$$

$$|a_{11}| \succ |a_{12}|$$

$$|a_{22}| \succ |a_{21}|$$

For n equations :

$$|aii| \succ \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|$$


x_2
v
x_1
u
(a)
x_2
v
x_1
u
(b)