CEN 3311 HEAT TRANSFER

UNSTEADY-STATE CONDUCTION

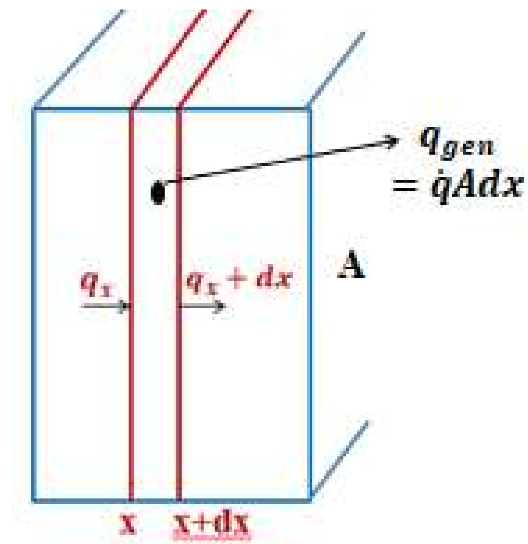
If the temperature of the solid is changing with time, this is unsteady state condition and the situation is more complex than steady state condition.

We will consider here the general case where the temperature may change with time and heat sources may be present within the body.

ENERGY BALANCE

Cartesian Coordinates:

For the element of thickness of dx, the following energy balance may be made:



$$\begin{bmatrix} Rate\ of\ energy \\ conducted\ into \\ the\ elemet \end{bmatrix} + \begin{bmatrix} Rate\ of\ energy \\ generated\ inside \\ the\ element \end{bmatrix} =$$

$$\begin{bmatrix} Rate\ of\ energy \\ conducted\ out \\ off\ the\ element \end{bmatrix} + \begin{bmatrix} Rate\ of\ energy \\ accumulated \\ stored\ inside \\ the\ element \end{bmatrix}$$

Rate of energy conducted into the element: $q\Big|_x = -kA\frac{\partial T}{\partial x}$

Rate of energy generated within element: $q_{gen} = \dot{q}Adx$ **(Adx = dV)**

Rate of energy conducted out off the element: $q\Big|_{x+dx} = \left[-kA\frac{\partial T}{\partial x}\right]_{x+dx}$

$$= -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

Rate of energy stored inside the element: $\rho C_p A\frac{\partial T}{\partial t}dx$ $\rho dV = \rho Adx$

Taylor series expansion:

$$f(a) = f(a_0) + f'(a_0)(a - a_0) + f''(a_0)\frac{(a - a_0)^2}{2!} + \cdots$$

Linear part (first 2 terms)

$$f(a) = f(a_0) + f'(a_0)(a - a_0)$$

In our case:

$$f(a_0) = q\big|_x \qquad f(a) = q\big|_{x+dx}$$

$$q\big|_{x+dx} = q_x + \frac{\partial}{\partial x}(q\big|_x)(x + dx - x)$$

$$q\big|_{x+dx} = \left(-kA\frac{\partial T}{\partial x}\right) + \left(-A\frac{\partial}{\partial x}k\frac{\partial T}{\partial x}\right)dx$$

$$q\big|_{x+dx} = -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

Here: $\dot{q}$ → energy generated per unit volume, $\boldsymbol{{}^{W}/_{m^3}}$

C_p → specific heat of material, $\boldsymbol{{}^{J}/_{kg{}^\circ C}}$

ρ → density, $\boldsymbol{{}^{kg}/_{m^3}}$

Combining the relations above gives:

$$-kA\frac{\partial T}{\partial x} + \dot{q}Adx = \rho C_p A\frac{\partial T}{\partial t}dx - A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

$$\boldsymbol{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t}}$$

One-dimensional heat conduction equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho C\frac{\partial T}{\partial t}$$

Three-dimensional heat conduction equation

Cylindrical coordinates:

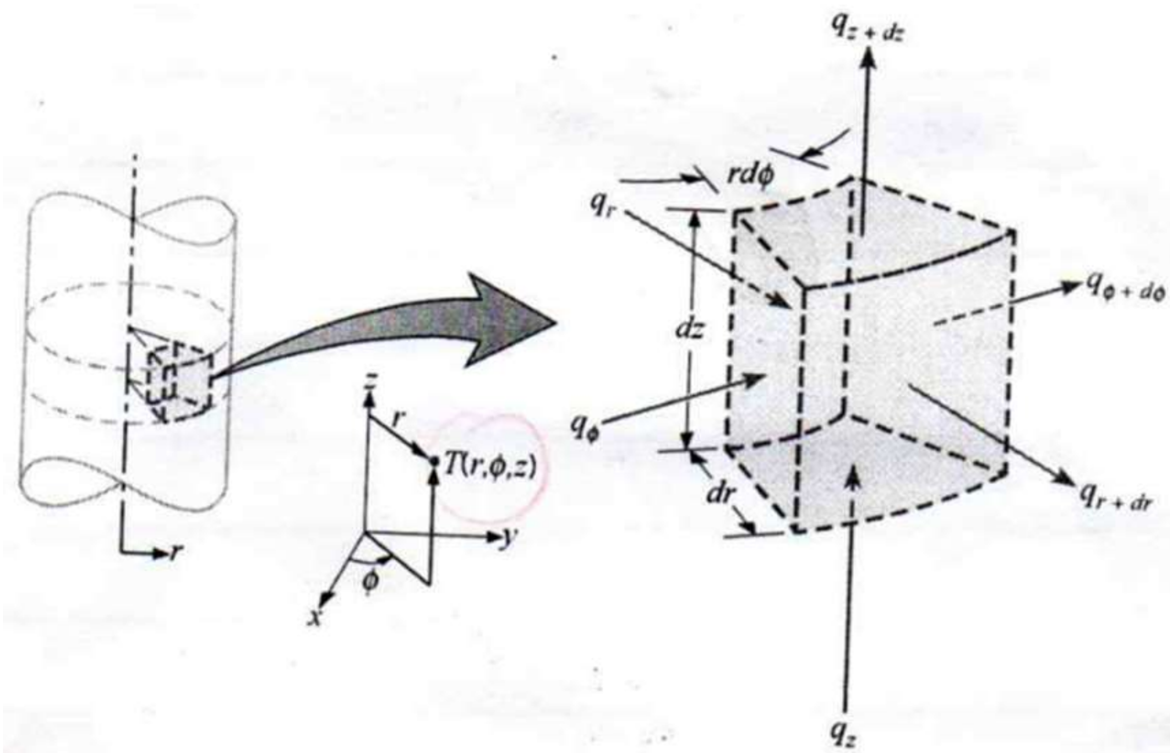


Figure Credit: Bergman, Lavine (2017)
Fundamentals of Heat and Mass Transfer, 8th Ed.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\propto}\frac{\partial T}{\partial t}$$

Spherical coordinates:

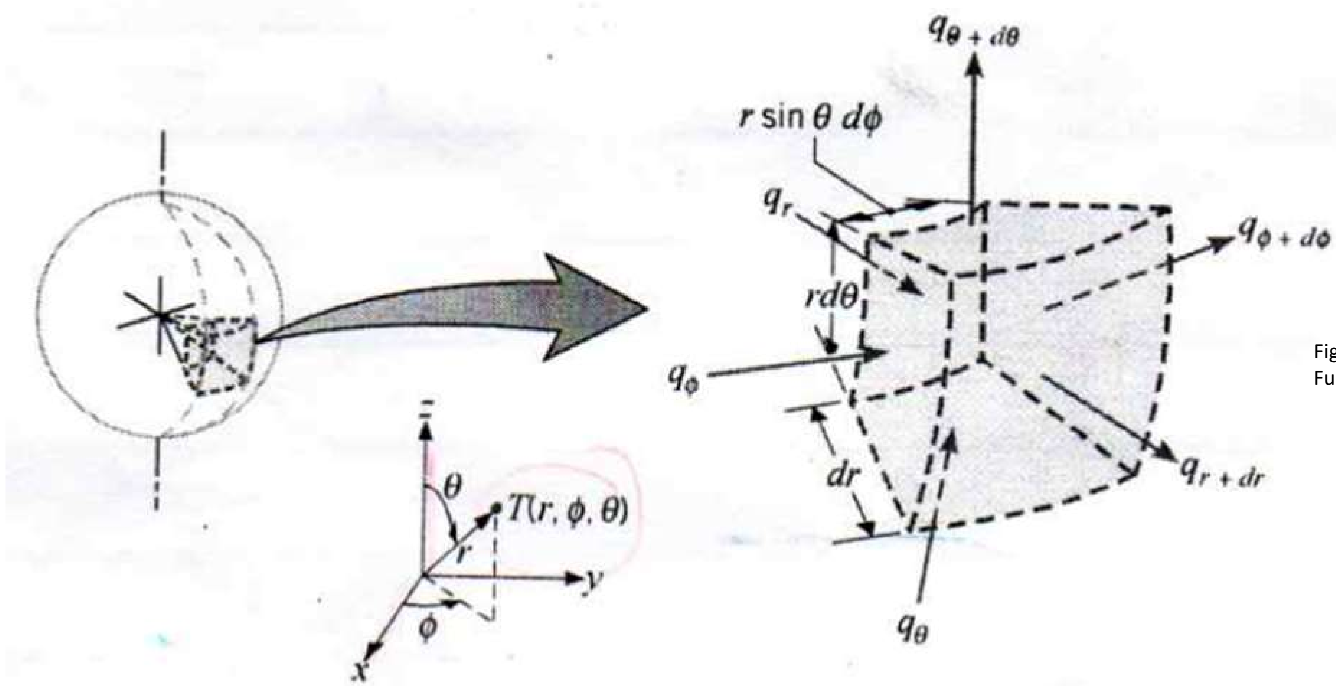


Figure Credit: Bergman, Lavine (2017)
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$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT) + \frac{1}{r^2 \sin\theta}\frac{1}{\partial\theta}(\sin\theta\,\frac{\partial T}{\partial\theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 T}{\partial \emptyset^2} + \frac{\dot{q}}{k} = \frac{1}{\propto}\frac{\partial T}{\partial t}$$