Quantum Mechanics

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Lets consider an example: In this example we will find eigenvectors and eigenvalues of $\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x$ operators on system which angular momentum $l = 1$.
Lets start the problem:

$$\hat{L}_x\hat{L}_y+\hat{L}_y\hat{L}_x = i\hbar L_z+2\hat{L}_y\hat{L}_x = i\hbar^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} +2\frac{L_+ - L_-}{2i}\frac{L_+ + L_-}{2}$$

$$= i\hbar^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \frac{\hbar^2}{2i} \begin{vmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{vmatrix} \begin{vmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{vmatrix} =$$

$$= i\hbar^2 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} - i\hbar^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{vmatrix} = i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$\boxed{\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x = i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}}$$

Lets find out what happens when this complex operators acts on states $1,1$, $1,0$ and $1,[-1$.

$$\left[\hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_x\right] 1,1 = i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = i\hbar^2 \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = i\hbar^2 1,-1$$

$$\left[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right]1,0 = i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} = -i\hbar^2 = 0$$

$$\left[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right]1,-1 = i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = -i\hbar^2 \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = -i\hbar^2 1,1$$

Finally, when this operator acts on states:

$$\left[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right] l, m = i\hbar^2 m l, -m$$

Now lets find the eigenvalues of this operator: First we need to find determinant of operator matrix
$det\left(\left[\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x\right] - \lambda I\right) = -\lambda(\lambda^2) - i\hbar^2(i\hbar^2\lambda) = -\lambda^3 + \hbar^4\lambda.$
Then $\lambda_0 = 0$,$\lambda_1 = \hbar^2$ and $\lambda^2 = -\hbar^2$

Lets construct the states:

$$i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = \hbar^2 \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

From this relation we find that, $-i\hbar^2 c = \hbar^2 a$ and $i\hbar^2 a = \hbar^2 c$, respectively $b = 0$,$a = i$ and $c = -1$ so:

$$\psi_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} i \\ 0 \\ -1 \end{vmatrix}$$

The second state:

$$i\hbar^2 \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix} = -\hbar^2 \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

$-i\hbar^2 c = -\hbar^2 a$ and $i\hbar^2 a = -\hbar^2 c$, respectively $b = 0$,$a = i$ and $c = 1$ so:

$$\psi_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} i \\ 0 \\ 1 \end{vmatrix}$$